Journal of Applied Geophysics 122 (2015) 62–73

Contents lists available at ScienceDirect

Journal of Applied Geophysics

journal homepage: www.elsevier.com/locate/jappgeo

Application of the least-squares inversion method: Fourier series versus waveform inversion

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ARTICLE INFO

Article history:
Received 14 January 2015
Received in revised form 7 August 2015
Accepted 21 August 2015
Available online 1 September 2015

Keywords:
Full waveform inversion
Fourier series
Least-squares inversion method

ABSTRACT

We describe an implicit link between waveform inversion and Fourier series based on inversion methods such as gradient, Gauss–Newton, and full Newton methods. Fourier series have been widely used as a basic concept in studies on seismic data interpretation, and their coefficients are obtained in the classical Fourier analysis. We show that Fourier coefficients can also be obtained by inversion algorithms, and compare the method to seismic waveform inversion algorithms. In that case, Fourier coefficients correspond to model parameters (velocities, density or elastic constants), whereas cosine and sine functions correspond to components of the Jacobian matrix, that is, partial derivative wavefields in seismic inversion. In the classical Fourier analysis, optimal coefficients are determined by the sensitivity of a given function to sine and cosine functions. In the inversion method for Fourier series, Fourier coefficients are obtained by measuring the sensitivity of residuals between given functions and test functions (defined as the sum of weighted cosine and sine functions) to cosine and sine functions. The orthogonal property of cosine and sine functions makes the full or approximate Hessian matrix become a diagonal matrix in the inversion for Fourier series. In seismic waveform inversion, the Hessian matrix may or may not be a diagonal matrix, because partial derivative wavefields correlate with each other to some extent, making them semi-orthogonal. At the high-frequency limits, however, the Hessian matrix can be approximated by either a diagonal matrix or a diagonally-dominant matrix. Since we usually deal with relatively low frequencies in seismic waveform inversion, it is not diagonally dominant and thus it is prohibitively expensive to compute the full or approximate Hessian matrix. By interpreting Fourier series with the inversion algorithms, we note that the Fourier series can be computed at an iteration step using any inversion algorithms such as the gradient, full-Newton, and Gauss–Newton methods similar to waveform inversion.

1. Introduction

The least-squares method, used to solve over-determined systems, has been commonly applied as a method of fitting data, and it is generally referred to as “regression” in statistics. One of the major applications of the least-squares method is to interpret subsurface structures from field data in geophysics, which is accomplished by minimizing a target objective function expressed as the sum of squared residuals between observed and modeled data. Geophysicists have made efforts to develop an efficient and accurate inversion method by exploiting the least-squares method. A variety of inversion methods were developed in all fields of geophysics. Nonetheless, seismic waveform inversion is still challenging because it generally suffers from the non-uniqueness problem of solutions, local minimum problems, and a large amount of data. Since Lailly (1983) and Tarantola (1984) introduced that a back-propagation technique of the reverse-time migration can be applied to seismic waveform inversion, a number of evolved versions of waveform inversion algorithms have emerged. Some of them are carried out in the time domain (Gauthier et al., 1986; Kolb et al., 1986; Mora, 1987), and others are performed in the frequency (Choi et al., 2008a,b; Pratt, 1999; Pratt and Shipp, 1999; Pratt et al., 1998; Shin and Min, 2006) or Laplace domains (Shin and Cha, 2008, 2009). Scales et al. (1990) suggested a regularization method, and Brandsberg-Dahl et al. (2003) applied a conjugate gradient method to seismic inversion method. Bunks et al. (1995) suggested a multiscale method to overcome a local minimum problem, and Sirgue and Pratt (2004) proposed a frequency-selection strategy for an efficient inversion algorithm. Crase et al. (1990) and Ravaut et al. (2004) applied the inversion algorithm to real field data. With the development of computational technology, the 3-D acoustic waveform inversion is becoming possible (Ben-Hadj-Ali et al, 2008; Sirgue et al., 2008; Vigh and Starr, 2008). However, the 3-D elastic waveform inversion is still challenging. Most of the aforementioned studies are based on the gradient, Gauss–Newton, or full Newton methods. Strictly speaking, only the Gauss–Newton and full Newton methods belong to the least-squares inversion methods.

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http://dx.doi.org/10.1016/j.jappgeo.2015.08.006
0926-9851/

In this study, we apply the common inversion methods to obtain Fourier coefficients, and compare the inversion process for Fourier analysis to that of seismic waveform inversion. By doing so, we can interpret the mathematical meanings of the classical Fourier analysis and seismic waveform inversion.

Fourier series have been used as a basic concept in studies on seismic data interpretation. Fourier analysis is based on the fact that any function can be expressed by the sum of weighted cosine and sine functions. Fourier coefficients are the constants that fit a sum of cosine and sine functions to a given function perfectly. Such coefficients have usually been obtained via classical Fourier analysis. In other words, the coefficients have been obtained from the orthogonal property of cosine or sine functions, which is done by integrating the production of the given function with cosine or sine functions for a given finite or infinite interval.

When we employ the conventional waveform inversion methods such as the gradient, full Newton, or Gauss–Newton methods to obtain the Fourier coefficients, we measure how the residuals between given functions and test functions (composed of cosine and sine functions) are sensitive to cosine or sine functions. Though seismic data are highly nonlinear and thus seismic waveform inversion is extremely complicated, the Fourier series are simple enough to be interpreted by using an inversion algorithm.

In the following sections, we first describe the original interpretation of Fourier analysis, and then we interpret Fourier series on the basis of seismic inversion algorithms. Next we review a forward modeling of wave equations using existing numerical modeling techniques such as the finite-difference or finite-element method, and the numerical structure of the resulting complex impedance matrix. Finally, we compare the Jacobian and Hessian matrices for Fourier series with those of seismic waveform inversion.

2. Fourier series

A Fourier series can be used to describe a function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \tag{1}$$

where the coefficients a_0, a_n, and b_n can be expressed by the definite integrals of the given function $f(x)$. If we compute the integral of Eq. (1) for an interval $[-\pi, \pi]$ and multiply it by $1/2\pi$, then we obtain (e.g., Boas, 1983; Kreyszig, 2006):

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} dx + \frac{a_1}{2\pi}\int_{-\pi}^{\pi}\cos x dx + \frac{a_2}{2\pi}\int_{-\pi}^{\pi}\cos 2x dx + \cdots \\ + \frac{a_n}{2\pi}\int_{-\pi}^{\pi}\cos nx dx + \frac{b_1}{2\pi}\int_{-\pi}^{\pi}\sin x dx + \cdots \\ + \frac{b_n}{2\pi}\int_{-\pi}^{\pi}\sin nx dx = \frac{a_0}{2}.\end{aligned} \tag{2}$$

Arranging Eq. (2) for the coefficient a_0 yields

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx. \tag{3}$$

In order to find the coefficient a_n, we compute

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x dx &= \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos nx dx + \frac{a_1}{2\pi}\int_{-\pi}^{\pi}\cos x \cos nx dx \\ &+ \frac{a_2}{2\pi}\int_{-\pi}^{\pi}\cos 2x \cos nx dx + \cdots + \frac{a_n}{2\pi}\int_{-\pi}^{\pi}\cos nx \cos nx dx \\ &+ \frac{b_1}{2\pi}\int_{-\pi}^{\pi}\sin x \cos nx dx + \cdots + \frac{b_n}{2\pi}\int_{-\pi}^{\pi}\sin nx \cos nx dx \\ &= \frac{a_n}{2\pi}\int_{-\pi}^{\pi}\cos nx \cos nx dx = \frac{1}{2}a_n.\end{aligned} \tag{4}$$

The coefficient a_n can be obtained from Eq. (4) as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx. \tag{5}$$

In the same manner, we can find b_n as

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx. \tag{6}$$

3. Interpretation of Fourier series in terms of inversion

The Fourier coefficients can also be computed by introducing an inversion method. When we compare the inversion for Fourier analysis with seismic waveform inversion using gradient methods, the coefficients a_n and b_n correspond to model parameters such as velocity, density, and elastic constants. If we define a test function as the sum of weighted cosine and sine functions, the objective function can be expressed by the l_2 norm of residuals between the given and test functions of

$$\begin{aligned}E &= \int_{-\pi}^{\pi}[u(x) - f(x)]^2 dx \\ &= \int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx - f(x)\right]^2 dx,\end{aligned} \tag{7}$$

where the test and given functions, $u(x)$ and $f(x)$, correspond to modeled data and field data in seismic waveform inversion, respectively. A set of optimal coefficients can be determined by minimizing the sum of residuals expressed by Eq. (7), which is done by calculating the gradients of the objective function with respect to the coefficients. The gradients are expressed as

$$\frac{\partial E}{\partial a_0} = \int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx - f(x)\right]dx, \tag{8}$$

$$\frac{\partial E}{\partial a_n} = \int_{-\pi}^{\pi} 2\cos nx\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx - f(x)\right]dx, \tag{9}$$

$$\frac{\partial E}{\partial b_n} = \int_{-\pi}^{\pi} 2\sin nx\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx - f(x)\right]dx. \tag{10}$$

From Eqs. (8)–(10), we note that computing the gradients is equivalent to measuring the sensitivity of residuals to sine or cosine functions.

In the full Newton method (e.g., Lines and Treitel, 1984), we scale the gradients with the Hessians. The Hessians are written analytically as

$$\frac{\partial^2 E}{\partial a_0^2} = \pi, \tag{11}$$

$$\frac{\partial^2 E}{\partial a_n^2} = \int_{-\pi}^{\pi} 2\cos^2 nx dx = 2\pi, \tag{12}$$

$$\frac{\partial^2 E}{\partial b_n^2} = \int_{-\pi}^{\pi} 2\sin^2 nx dx = 2\pi, \tag{13}$$

$$\frac{\partial^2 E}{\partial a_m \partial a_n} = 0 \quad \text{for} \quad m \neq n, \tag{14}$$