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Research Paper

# Non-probabilistic set-theoretic models for transient heat conduction of thermal protection systems with uncertain parameters

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## HIGHLIGHTS

- Interval analysis method is utilized for transient heat conduction.
- Convex models are utilized for transient heat conduction.
- In convex models, we propose a novel convex model to quantify uncertain parameters.
- The novel convex model can partly reduce the space of temperature field response.

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## ABSTRACT

Thermal protection systems (TPS) play a key role in the development of hypersonic aircrafts and the performance of TPS is directly in connection with its temperature field, thus a number of analytical and experimental studies have been conducted to study heat transfer analysis. Due to the existence of uncertain parameters in the temperature field, it is imperative to adopt the approaches involving uncertainty analysis to obtain reliable results. The non-probabilistic set-theoretic models, compared with the probabilistic approach, only require a small amount of experimental samples to process the study of uncertainties. Interval analysis method (IAM), classical convex model (CCM) and novel convex model (NCM) are applied to quantify uncertain parameters in TPS and then combined with finite elemental differential equation of transient thermal analysis to study the effects of uncertain parameters on temperature field response by means of Taylor series expansion. Moreover, the thermal responsive bounds in both CCM and NCM are yielded by the Lagrange multiplier method. A ceramic TPS is performed to illustrate the application of the present method and the results show that NCM can reduce the space of temperature field responses. Besides, the non-probabilistic set-theoretic methods can serve for the design of TPS.


## 1. Introduction

During the atmospheric flight, the hypersonic aircraft would experience significant aerodynamic heating due to its hypersonic speed. In order to ensure the internal structures within the operating temperature range, TPS should be employed to achieve the heat insulation effect. The strength and stability of TPS are directly in connection with its temperature field, thus the thermal response analysis of TPS has always been a hot research topic [1].

Deterministic heat transfer analysis approaches neglect the significant effects of uncertain parameters on heat transfer analysis. However, there exist many uncertainties associated with material properties, physical dimensions, exterior loads, and environmental factors in the engineering practice [2]. Appropriate models have to be utilized to deal with these uncertainties. Determining the effects of the uncertainties becomes essential in temperature field analysis and the uncertainty of temperature responses is very important to the design of TPSs in which even a few degrees of temperature fluctuation may cause serious problems.

Many different approaches can be applied to uncertain propagation and quantification and generally classified into three categories: probabilistic approach, fuzzy approach and set-theoretical approach [3]. Nowadays, the probabilistic approach, such as the Monte Carlo method [4,5], the perturbation stochastic method [6,7], the spectral stochastic method [8,9], and the other methods [10,11], is designated as the most valuable tools for dealing with uncertain problems in which the uncertain parameters are regarded as random variables. Unfortunately, there is always not enough experimental data to describe and determine the probability distribution of uncertainties in practical engineering. Thus the application of the probabilistic approach has been confined to the above situation [12]. Consequently, much of the research in the last two decades has been focusing on set-theoretical approach for uncertainty analysis. According to the different quantification of uncertain

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parameters, the set-theoretical approach can be categorized into two groups: interval analysis method and convex model method. In the interval method, the uncertain parameters are assumed as interval variables belonging to a hyper-rectangle. In the convex method, the uncertain parameters are assumed as convex variables belonging to an ellipsoid. In recent years, there have been a large number of research results regarding the set-theoretical methods [2,3,12–20]. The set-theoretical methods have been successfully applied to the static displacement, dynamic response and structural reliability of the engineering structure with uncertain-but-bounded parameters. However, from the overall perspective, the research on the numerical analysis of the uncertain problem with uncertain-but-bounded parameters is still in its preliminary stage and some paramount issues still remain unsolved. For example, the application of set-theoretical methods in the response analysis of thermal field with uncertain parameters needs to be exploited.

In addition, the classical convex model is widely used and generally derived from an ellipsoid enclosing the rectangle containing all experimental samples [20]. However, the classical convex model may not be the optimal/smallest one and the results obtained by the classical convex model are always conservative on the structural responses, especially some experimental points are nearly located at the middle points of the edges of the rectangle, which leads to increasing manufacturing costs.

As is discussed above, the purpose of this paper is to present the set-theoretical methods (namely IAM, CCM and NCM) to quantify the uncertain parameters and then combine them with finite element differential equation of transient thermal analysis to study the effects of uncertain parameters on temperature responses by means of Taylor series expansion. Moreover, the thermal responsive bounds in both CCM and NCM are yielded by the Lagrange multiplier method.

## 2. Transient heat transfer problem of thermal protection systems with uncertain-but-bounded parameters

The governing differential equation of transient thermal analysis in finite element method (FEM) can be represented as follows

$$\boldsymbol{C}\dot{\boldsymbol{T}}(t)+\boldsymbol{K}\boldsymbol{T}(t)=\boldsymbol{Q}(t) \tag{1}$$

where, $\boldsymbol{K}$ is the conductance matrix, $\boldsymbol{C}$ is the specific heat matrix, $\boldsymbol{T}$ is the unknown nodal temperature vector, $\boldsymbol{Q}$ is the vector of the nodal heat flow, and a dot over a variable denotes differentiation with respect to time $t$. The physical, material, exterior loads and geometric properties in TPS are called structural parameters. The structural conductance matrix, specific heat matrix and vector of the nodal heat flow are all functions of structural parameters.

In engineering practice, uncertain parameters are often temperature-independent variables, and are sometimes temperature-dependent variables, especially for hypersonic aircrafts. In this section we firstly consider the case in which transient heat transfer problem with temperature-dependent parameters is described with the use of set-theoretic approach. An approximate linear relationship, instead of nonlinear relationship, is always utilized in thermal analysis with temperature-dependent parameters by means of FEM. That is to say, a series of polylines take the place of temperature-dependent curves. Thus the key process of transient heat conduction is to determine the information about points on the curves of temperature-dependent structural parameters.

Consider now a situation in which some of the points on the curves of temperature-dependent structural parameters are uncertain variables and suppose that $a_i(T_{ij})$ denotes the $i$-th structural parameter value at $T_{ij}$ degrees centigrade. By introducing the temperature-dependent uncertainties $\boldsymbol{a}(T_j)=(a_i(T_{ij}))_m$, Eq. (1) can be rewritten as

$$\boldsymbol{C}(\boldsymbol{a}(T_j))\dot{\boldsymbol{T}}(t)+\boldsymbol{K}(\boldsymbol{a}(T_j))\boldsymbol{T}(t)=\boldsymbol{Q}(\boldsymbol{a}(T_j),t) \tag{2}$$

Obviously, the nodal temperature vector $\boldsymbol{T}$ is also a function of the temperature-dependent uncertainties $\boldsymbol{a}(T_j)$ and can be assumed as

$$\boldsymbol{T}=\boldsymbol{T}(\boldsymbol{a}(T_j),t)=\boldsymbol{T}\left((a_i(T_{ij}))_m,t\right) \tag{3}$$

According to the idea of the set-theoretic approach, the temperature-dependent uncertainties $\boldsymbol{a}(T_j)$ belong to a domain $\boldsymbol{\Gamma}$ (such as interval set $\boldsymbol{\Gamma}_{\mathrm{I}}$ or convex set $\boldsymbol{\Gamma}_{\mathrm{C}}$). The theoretical solution set of the nodal temperature vector $\boldsymbol{T}(\boldsymbol{a}(T_j))$ can be obtained by solving Eq. (2) in which the temperature-dependent uncertain vector $\boldsymbol{a}(T_j)$ assumes all possible values inside the domain $\boldsymbol{\Gamma}$. Thus, the theoretical solution set of the nodal temperature vector $\boldsymbol{T}(\boldsymbol{a}(T_j))$ can be expressed as

$$\begin{aligned}\boldsymbol{T}(a)=\{&\boldsymbol{T}(\boldsymbol{a}(T_j))|\boldsymbol{C}(\boldsymbol{a}(T_j))\dot{\boldsymbol{T}}(\boldsymbol{a}(T_j),t)+\boldsymbol{K}(\boldsymbol{a}(T_j))\boldsymbol{T}(\boldsymbol{a}(T_j),t)\\&=\boldsymbol{Q}(\boldsymbol{a}(T_j),t),\quad \boldsymbol{a}\in\boldsymbol{\Gamma}\}\end{aligned} \tag{4}$$

The problem is to determine the narrowest interval $\boldsymbol{T}^{\mathrm{I}}(\boldsymbol{a})$ containing all possible response vector, namely, find the upper and lower bounds of the nodal temperature vector

$$\begin{aligned}\boldsymbol{T}_{\mathrm{upper}}&=\max_{\boldsymbol{a}(T_j)\in\boldsymbol{\Gamma}}\{\boldsymbol{T}(\boldsymbol{a})|\boldsymbol{C}(\boldsymbol{a}(T_j))\dot{\boldsymbol{T}}(\boldsymbol{a}(T_j),t)+\boldsymbol{K}(\boldsymbol{a}(T_j))\boldsymbol{T}(\boldsymbol{a}(T_j),t)=\boldsymbol{Q}(\boldsymbol{a}(T_j),t)\}\\ \boldsymbol{T}_{\mathrm{lower}}&=\min_{\boldsymbol{a}(T_j)\in\boldsymbol{\Gamma}}\{\boldsymbol{T}(\boldsymbol{a})|\boldsymbol{C}(\boldsymbol{a}(T_j))\dot{\boldsymbol{T}}(\boldsymbol{a}(T_j),t)+\boldsymbol{K}(\boldsymbol{a}(T_j))\boldsymbol{T}(\boldsymbol{a}(T_j),t)=\boldsymbol{Q}(\boldsymbol{a}(T_j),t)\}\end{aligned} \tag{5}$$

## 3. Interval analysis method for transient thermal analysis with uncertain parameters

In terms of the center point and the radius of an interval vector in interval analysis, the nominal vector of uncertain parameters can be written as

$$\boldsymbol{a}^0(T_j)=(a_i^0(T_{ij}))_m=\frac{1}{2}\left(\max_{1\le r\le M}\boldsymbol{a}^{(r)}(T_j)+\min_{1\le r\le M}\boldsymbol{a}^{(r)}(T_j)\right) \tag{6}$$

where $M$ is the number of experimental samples.

And the radius vector of uncertain parameters can be denoted as

$$\Delta\boldsymbol{a}(T_j)=(\Delta a_i(T_{ij}))_m=\frac{1}{2}\left(\max_{1\le r\le M}\boldsymbol{a}^{(r)}(T_j)-\min_{1\le r\le M}\boldsymbol{a}^{(r)}(T_j)\right) \tag{7}$$

Thus, the uncertain parameters can be represented in the following form

$$\boldsymbol{a}(T_j)=\boldsymbol{a}^0(T_j)+\boldsymbol{\delta}(T_j),\quad \boldsymbol{\delta}(T_j)\in\Delta\boldsymbol{a}^{\mathrm{I}}(T_j)=[-\Delta\boldsymbol{a}(T_j)\quad \Delta\boldsymbol{a}(T_j)] \tag{8}$$

Based on the first-order Taylor expansion, the nodal temperature vector $\boldsymbol{T}(\boldsymbol{a}(T_j))$ can be approximated as

$$\boldsymbol{T}(\boldsymbol{a}(T_j))=\boldsymbol{T}(\boldsymbol{a}^0(T_j)+\boldsymbol{\delta}(T_j),t)=\boldsymbol{T}(\boldsymbol{a}^0(T_j),t)+\sum_{i=1}^{m}\frac{\partial\boldsymbol{T}(\boldsymbol{a}^0(T_j),t)}{\partial a_i(T_{ij})}\cdot\delta(T_{ij}) \tag{9}$$

The $k$-th element $T_k(\boldsymbol{a}(T_j))$ in the nodal temperature vector $\boldsymbol{T}(\boldsymbol{a}(T_j))$ can be expressed as

$$T_k(\boldsymbol{a}(T_j))=T_k(\boldsymbol{a}^0(T_j)+\boldsymbol{\delta}(T_j),t)=T_k(\boldsymbol{a}^0(T_j),t)+\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\delta}(T_j) \tag{10}$$

where

$$\boldsymbol{g}^{\mathrm{T}}=\left[\frac{\partial T_k(\boldsymbol{a}^0(T_j),t)}{\partial a_1(T_{ij})}\quad\frac{\partial T_k(\boldsymbol{a}^0(T_j),t)}{\partial a_2(T_{ij})}\quad\cdots\quad\frac{\partial T_k(\boldsymbol{a}^0(T_j),t)}{\partial a_m(T_{ij})}\right] \tag{11}$$