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Research paper

Proposing a normalized Biot number: For simpler determination of Fourier exponents and for evaluating the sensitivity of the Biot number

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HIGHLIGHTS

- A normalized Biot number [Bi_{norm}] has been formulated.
- Enabling simple prediction of Fourier exponents and lag-factors.
- Easy assessment of sensitivity of the Fourier expansion.
- Graphical and tabulated misreads is eliminated.

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ABSTRACT

This paper presents a normalization of the Biot number, which enables the Fourier exponents to be fitted with a simple 3rd order polynomial ($R^2 > 0.9999$). The method is validated for Biot numbers ranging from 0.02 to ∞, and presented graphically for both the Fourier exponents and the lag factors needed in the series expansion. The lag factors and Fourier exponents are validated with an average variation coefficient (CVRMSD) less than 0.006. The resulting prediction error of the thermal response is < 0.6 °C for spheres and <0.3 °C for slabs and cylinders. The normalized Biot number also facilitates an easy investigation of the sensitivity in heat transfer calculations. The simplicity of the solution facilitates its implementation in the industry and curricula for engineers that needs crude calculation methods for thermal calculations, e.g. food science and food technology educations.

1. Introduction

The calculation of heating and cooling of solids is a traditional engineering task in several industries, also in food manufacture where the production of food is conducted in large scale factories often expanded from smaller productions, based on a history of trial and error. To ensure safe products and to evaluate the quality of manufactured food, it is crucial to know the thermal history that the products undergo during processing, cooling, storage and distribution. The staff at most current food production sites often do not have an advanced engineering training, and they must rely on simple, robust calculations in their daily process evaluation and performing scheduling activities. This paper is proposing an easy way to evaluate thermal response based on a normalized Biot number, suited for such calculations.

For evaluation of the thermal response in solid foods the series expansion to the Fourier equation for transient conduction heat transfer is the standard procedure for food products which approximately can be described by ideal geometries (infinite slab, infinite cylinder and sphere) and cross-sections of the first two.

The mathematical apparatus for transient conduction heat transfer using the series expansion is thoroughly described by Carslaw and Jaeger [1]. It is the standard for predictive heat transfer calculations, and it is presented in most textbooks on the subject [2,3]. The series expansion is presented below in condensed form for ideal geometries (infinite slabs, infinite cylinders and spheres).

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Nomenclature

Bi	Biot number $Bi = \frac{h}{k} \cdot L$ [-]
Bi_{norm}	Normalized Biot number $Bi = \frac{Bi}{Bi+1}$ [-]
$a_{c,1}$	Lag factor (center temperatures) in the series expansion [-]
$a_{m,1}$	Lag factor (average temperatures) in the series expansion [-]
b_i	Fourier exponent $b_i = \lambda_i^2$ [-]
c_p	Specific heat capacity [J/kg·K]
α	Thermal diffusivity $\alpha = \frac{k}{\rho \cdot c_p}$ [m^2/s]
h	Heat transfer coefficient [W/m^2 K]
T	Time [s]
J_0	0^{th} order of the Bessel function of the first kind
J_1	1^{st} order of the Bessel function of the first kind
k	Thermal conductivity [W/m^2]
Ω	Dimensionless temperature difference $\Omega = \frac{(T_s - T)}{(T_s - T_0)}$ [-], subscripts *s* is the surrounding temperature, *0* is the initial temperature
ρ	Density [kg/m^3]
λ_1	The first eigenvalue to respective root functions [-]
R	Determining dimension (1/2 height for slabs, radius for cylinders and spheres) [m]
Fo	Fourier number, dimensionless process time $Fo = \frac{\alpha}{R^2} \cdot t$ [-]

The series expansion for heat transfer:

$$\Omega_k = \left(\frac{T_s - T}{T_s - T_0} \right) = \sum_{i,k}^{\infty} a_{i,k} e^{-b_i \cdot Fo} \qquad [1]$$

Where k = x/R for point temperatures, k = c for center temperatures and k = m for volume average temperatures, the derived values for the lag factors $a_{i,k}$ and Fourier exponents b_i is calculated from the expressions in Table 1.

The Fourier number (Fo):

$$Fo = \frac{\alpha}{R^2} \cdot t \qquad [2]$$

The Fourier exponent b_i in Equation (1) is calculated from the eigenvalue (λ_i) to the respective root functions:

$$b_i = \lambda_i^2 \qquad [3]$$

The eigenvalues are calculated by iteration from the root functions in Table 1 based on the Biot number. The equations for the derived Fourier exponents (b_i) and lag factors ($a_{i,k}$) are presented along in Table 1.

Table 1
Mathematical presentation of the respective root functions for the ideal geometries, lag factors for center temperatures (a_c), the positional lag factors ($a_{x/R}$) and the lag factors for mean temperatures (a_m).

Geometry	Root function λ_i	$a_{c,i}$	$a_{x/R,i}$	$a_{m,i}$
Inf. Plate	$Bi = \lambda_i tan\lambda_i$	$\frac{2sin\lambda_i}{\lambda_i + sin\lambda_i cos\lambda_i}$	$a_c \cdot \cos\left(\lambda_i \frac{x}{L}\right)$	$a_c \cdot \frac{sin(\lambda_i)}{\lambda_i}$
Inf. Cylinder	$Bi = \frac{\lambda_i J_1(\lambda_i)}{J_0(\lambda_i)}$	$\frac{2J_1(\lambda_1)}{\lambda_i (J_0^2(\lambda_i) + J_1^2(\lambda_i))}$	$a_c \cdot J_0\left(\lambda_i \frac{r}{R}\right)$	$a_c \cdot 2\frac{J_1 \lambda_i}{\lambda_i}$
Sphere	$Bi = 1 - \lambda_i cot\lambda_i$	$\frac{2(sin\lambda_i - \lambda_i cos\lambda_i)}{\lambda_i - sin\lambda_i cos\lambda_i}$	$a_c \cdot \frac{sin\left[\lambda_i \left(\frac{r}{R}\right)\right]}{\lambda_i \left(\frac{r}{R}\right)}$	$a_c \cdot 3\frac{sin(\lambda_i) - \lambda_i cos(\lambda_i)}{\lambda_i^3}$

J_0 and J_1 is the Bessel function of the 1st kind with 0^{th} and 1st order respectively.

The Biot number (Bi) is the ratio between the external and internal resistance to heat transfer and is calculated using Equation (4):

$$Bi = \frac{h}{k} R \qquad [4]$$

Where *h* is the heat transfer coefficient, *k* is the thermal conductivity and *R* is the characteristic dimension. In the simplest form, and also throughout this paper the convective heat transfer coefficient *h* is considered constant in time and the same for all position on the surface of the body.

The lag factor *a* can be calculated from the equations in Table 1, where index c denotes the center temperature, index *x/r* denotes a position relative to the center and *m* denotes the volume average temperature.

The developed series applies for ideal geometries that can be described in simple coordinate systems (the infinite slab in Cartesian coordinates, the infinite cylinder in cylindrical coordinates and spheres in a spherical coordinate system). For calculating geometries that can be expressed as the cross section of ideal geometries [4] the thermal response can be calculated from Equation (5), here exemplified by the calculation of a can-shaped geometry.

$$\frac{(T_s - T)}{(T_s - T_0)} = \Omega_{can} = \Omega_{\frac{1}{2}height} \times \Omega_{radius} \qquad [5]$$

In the early-mid 20th century heat transfer calculations were time-consuming to conduct without the availability of computers; thus several graphical methods and tabulated values for determining the Fourier exponents and lag factors have been constructed. They are presented also in recent standard textbooks on the subject [2,3]. For a fast evaluation of thermal response as a function of Bi and Fo several graphical methods have also been reported such as the Guernay-Lourie plots and the Heissler charts [5,6]. The series expansion solutions to non-stationary heat transfer have been thoroughly mathematically described [1], and they are presented in a more condensed format [7]. The solutions from Ref. [7] are still rendered in textbooks today [2,3]. Even though several more advanced tools and techniques (simulation software, and finite element calculations) are widely applied in research, they are rarely used in the food manufacturing industry or in teaching for several reasons: the software is expensive and the training needed for the employees or students in order to conduct and utilize the obtained information from these calculations is intensive.

In many processing situations it is often adequate to acquire information on the thermal response in the last part of the process. In these situations the dimensionless temperature difference will be low and the Fourier number will be fairly large (Fo > 0.2). For calculations where the Fourier number is above 0.2 the 1st term in the series expansion is assumed adequate for evaluating the thermal response [2]. Christensen and Adler-Nissen [8] showed that also the initial phase in heating and cooling of solids can be modelled by an extended 1st term approximation where the first eigenvalue is the only needed input parameter to cover $0<Fo\rightarrow\infty$.

One of the big challenges using the solutions devised by Ref. [7] is the determination of the Fourier exponents given by the eigenvalues to the respective root functions (Table 1). As mentioned, the root functions are of iterative character and are thus cumbersome to solve. Alternatively, the exponents can be found tabulated in textbooks or papers on the subject [3,7], where it is often needed to interpolate between tabulated values, or they can be found in charts [2] where there is a risk of misreads. Neither the tabulated values nor the graphical representation are suited for implementation in simple programs or spreadsheets. Thus, it would be a great advantage to develop non-iterative equations for calculating