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Research paper

# Simplified equations for transient heat transfer problems at low Fourier numbers

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## HIGHLIGHTS

- A simple analytical solution to heat transfer is proposed.
- Prediction of the thermal history ($0.1 < Bi < 100$, $0 < Fo < \infty$).
- Adaptation to an industrial case is presented.
- Validation with a general precision of RMSD < 0.01.

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## ABSTRACT

This paper proposes an analytical solution to transient heat transfer, which also applies for the initial heating/cooling period ($Fo < 0.2$) of solids with simple geometries subjected to convective boundary conditions, with negligible mass transfer and phase-change. The new equation is presented and validated for infinite slabs, infinite cylinders and spheres and by an industrial application example, covering the center temperature and the volume average temperature. The approach takes ground in the residual difference between a 1 term series solution and a 100 term solution to the Fourier equation of the thermal response for solids subjected to convective heat transfer. By representing the residual thermal response as a function of the Biot number and the first eigenvalue, the new approach enables the description of the thermal response in the whole Fourier regime. The presented equation is simple and analytical in form, which allows an easy implementation into spreadsheets and thus serves as a transparent and fast tool for crude process calculations in e.g. process planning or introduction of new products to existing lines. The prediction error of the new equation is low (RMSD < 0.015) for $0 < Fo < 0.2$ and $0.1 < Bi < 100$ for infinite slabs, infinite cylinders, spheres and typical examples of finite bodies.



## 1. Introduction

The calculation of non-stationary convective heat transfer into solids is important in several engineering fields, such as aeronautics, metallurgy, building construction and food technology [1–3]. For elementary geometries (infinite slab, infinite cylinder, sphere), the standard approach is to solve the Fourier differential equation through a series expansion [4,5]. The solution assumes convective uniform boundary conditions with no or insignificant mass transfer, no internal heat generation and negligible changes in geometry. Despite these restrictions, many practical engineering problems of convective heating or cooling of solids can be approximated by the situation with ideal geometries or cross-sections of them. In food engineering, calculation of the sterilization process in the canning industry is the classical example [6,7]. Also calculation of cooling processes is an often encountered issue in the food industry where it is important for the safety and quality of the food to ensure that a target temperature of a solid food product is reached before it enters a chilled storage.

In recent decades research in heat transfer, also within food engineering, has focused on modeling and simulations in advanced software such as the MATLAB based COMSOL Multiphysics® [8,9]. The simulations have the advantage that more complex physics can be included into the models accounting for mass transfer, geometry changes, chemical reactions and structure changes within the products [10]. An example of this modeling approach is a study of the convective roasting of meat [11], where heat and mass transfer is coupled with a geometrical change (shrinkage) during

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**Nomenclature**

| | |
|---|---|
| $\alpha_{Bi}$ | slope of regression curves |
| $a_i$ | lag factor in the series expansion to the heat transfer equation [−] |
| $a_c$ | lag factor center temperature |
| $a_m$ | lag factor volume average temperature |
| $Bi$ | Biot number $Bi = h/k \cdot L$ [−] |
| $c_p$ | specific heat capacity [J/kg K] |
| $C$ | intercept of regression curves |
| $\varepsilon$ | residual dimensionless temperature difference [−] |
| $Fo$ | Fourier number (dimensionless process time) $Fo = (k/(L^2 \cdot \rho \cdot c_p)) \cdot t$ [−] |
| $h$ | heat transfer coefficient [W/m$^2$ K] |
| $T$ | temperature [°C] |
| $J_0$ | 0th order of the Bessel function of the first kind [−] |
| $J_1$ | 1st order of the Bessel function of the first kind [−] |
| $k$ | thermal conductivity [W/m$^2$] |
| $\lambda_i$ | the eigenvalue to respective root functions [−] |
| $L$ | characteristic dimension [m] |
| $\Omega$ | dimensionless temperature difference $\Omega = (T_s - T)/(T_s - T_0)$ [−], subscripts $s$ is surrounding temperature, 0 is initial temperature |
| $\rho$ | density [kg/m$^3$] |
| $x$ | relative position in geometry, $x = 0$ for center, 1 for surface |
| $t$ | time [s] |

processing. Such simulations are precise and can handle real processing situations; however, they are often targeted at specific products in specific processes making them less versatile for general engineering calculations. In addition, many engineers in some industrial sectors (such as the food producing industry) do not have access to suitable software nor the time needed to conduct simulations in their daily work. This means that there is still a need for classical engineering equations to handle the calculation of the thermal history of solid foods, preferably by using simple, commonly used software such as spreadsheets.

Studies directed towards more simple engineering equations to handle low Fourier numbers are scarce, and they are neither widely implemented in the industry nor presented in textbooks. Hayakawa [12] reported charts and tables to estimate the initial center temperature response for canned foods, but the solutions are limited to the geometry of a finite cylinder. Ramaswamy and Shreekanth [13] used a stepwise multiple regression approach to approximate the summarized series solution at $Fo < 0.2$ for infinite geometries. Their general idea was to model the residual between the summarized series and the 1 term approximation, as is also done in the present study. Their solution included 13 new parameters for each of the three infinite geometries in a set of three equations [13]. As it will be shown, we propose a simpler solution of sufficient precision with only 1 new parameter included in a single equation for handling the prediction of the center temperature and the volume average temperature. The solution covers a wide application area ($0 < Fo$, $0.1 < Bi < 100$) for the elementary geometries and their cross-sections, cans, prisms, and boxes. The above described studies are connected to food processing where transient heat transfer is central.

For the calculation of thermal storage Xu et al. [14] extended the validity of the lumped capacitance equation for elementary geometries and for the geometry of a hollow cylinder. Their validation was most valid for high Fourier numbers. Based on an analogous approach to [14] Jian et al. [15] also suggested to use an extended lumped capacitance model for calculating the thermal storage in a hollow tube geometry. The theoretical validation [15] suggests a good fit also at $Fo < 0.2$ as long as the annulus in the hollow pipe is much smaller than the outer diameter.

## 2. Theory

For the infinite slab, the infinite cylinder, and the sphere the temperature history can be calculated using Equations (1)–(7) below [5], assuming convective boundary conditions with no or insignificant mass transfer and disregarding the influence of chemical reactions and possible changes in geometry.

$$\Omega = \sum_{i=1}^{\infty} a_i \cdot e^{-\lambda_i^2 \cdot Fo} \tag{1}$$

where ($Fo$) is the Fourier number describing the dimensionless time of a process duration. $Fo$ is calculated from the thermophysical properties of the material and the characteristic dimension. The lag factor ($a_i$) and the Fourier exponent ($\lambda^2$) is determined from the Biot number ($Bi$). $Bi$ is the ratio of internal vs external resistance to heat transfer and is a dimensionless representation of whether the external heat transfer or the internal conduction is dominating the process duration. If $Bi \rightarrow 0$ the process duration is only dominated by the external heat transfer coefficient, if $Bi \rightarrow$ infinity the process duration is only determined by the thermal conductivity of the material.

The lag factor ($a_i$) and the Fourier exponent ($\lambda^2$) is calculated from the eigenvalue ($\lambda_i$) to the representative root functions Equations (2) and (3) and 4, for the 3 elementary geometries.

$$\text{inf.slab} : Bi = \lambda_i \tan \lambda_i \tag{2}$$

$$\text{inf.cylinder} : Bi = \frac{\lambda_i J_1(\lambda_i)}{J_0(\lambda_i)} \tag{3}$$

$$\text{sphere} : Bi = 1 - \lambda_i \cot \lambda_i \tag{4}$$

The lag factor ($a_i$) is calculated for the three geometries as a function of the respective eigenvalue to the root functions in Equations (5)–(7). $L$ is the characteristic dimension, which is half thickness for slabs, and the radius for cylinders and spheres.

$$\text{inf.slab (point)} : a_i = \frac{2 \sin \lambda_i}{\lambda_i + \sin \lambda_i \cos \lambda_i} \cdot \cos\left(\lambda_i \frac{x}{L}\right) \tag{5}$$

$$\text{inf.cylinder (point)} : a_i = \frac{2J_1(\lambda_i)}{\lambda_i \left[J_0^2(\lambda_i) + J_1^2(\lambda_i)\right]} \cdot J_0\left(\lambda_i \frac{x}{L}\right) \tag{6}$$

$$\text{sphere (point)} : a_i = \frac{2(\sin \lambda_i - \lambda_i \cos \lambda_i)}{\lambda_i - \sin \lambda_i \cos \lambda_i} \cdot \frac{\sin\left(\lambda_i \frac{x}{L}\right)}{\lambda_i \frac{x}{L}} \tag{7}$$

For estimating the center temperature in finite bodies such as cans, boxes and prisms the cross-products of the elementary geometries is the standard solution [16]. The resulting dimensionless temperature difference ($\Omega$) is the product of the individual contributions, here exemplified by the calculations of a finite box:

$$\Omega_{box} = \Omega_{length} \cdot \Omega_{width} \cdot \Omega_{height} \tag{8}$$