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Bending and buckling of nonlocal strain gradient elastic beams

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ABSTRACT

Featured by the two material length parameters in the nonlocal strain gradient theory, it is still unknown that what are the boundary conditions of nonlocal strain gradient beams, since the equations of motion and boundary conditions of these beam models appear in the same form as those of the classical ones. Based on the weighted residual approaches, this paper provides the boundary value problems of Euler–Bernoulli beams within the framework of the nonlocal strain gradient theory in conjunction with the von Kármán nonlinear geometric relation. The closed-form solutions for bending and buckling loads of nonlocal strain gradient beams are obtained. Numerical results show that the higher-order boundary conditions have no effect on the static bending deflection of beams for the cases studied. However, the higher-order boundary conditions and the material length parameters have a significant effect on the buckling loads. Finally, when the two material length parameters are the same, the buckling loads can not always reduce to the classical solutions, the findings of which violate our expectations. The results provided in this work are expected to be helpful for the applications of this theory to the analysis of engineering structures.

1. Introduction

Engineering structures such as beams, plates and shells have been widely used in micro- and nano-sized sensors, actuators, atomic force microscopes. In these applications, size effects of material properties are observed at small sizes both in experimental works [1–4] and in numerical simulations [5–7]. The aforementioned works show that the materials exhibit either stiffening behaviors or softening behaviors in comparison to the bulk cases. Therefore, continuum theories that can capture the size effects of materials at small sizes have attracted considerable attention in the research communities with the view toward a better understanding and characterization of materials.

Based on the concept that the stress at a reference point is not only a function of the reference point, but also the strain at all points of the body, Eringen [8] developed an elasticity theory for the applications in surface waves. With the emerging of carbon nanotubes and graphene sheets, this theory have been extended to the study of the static and dynamic behaviors of structures in terms of rods [9–14], beams [14–25], plates [26–33] and shells [34–38]. For more details, the interested reader may refer to the recent reviews by Arash and Wang [39] and Eltaher, et al. [40].

In general, the use of this theory results in the softening effect when it is compared with the classical elasticity theory. However, two issues violate the softening phenomena. The first issue is that the bending solutions of nonlocal models in some cases are found to be the same as the classical solutions. In other words, the size effects vanish for cantilever beams subjected to concentrated forces [41]. To address this issue, Challamel and Wang [42] proposed a gradient elastic model as well as an integral nonlocal elastic model that is based on combining the local and the nonlocal curvatures in the constitutive relation. After this, several fresh ideas are raised to clarify this issue [16,43–46]. For example, Khodabakhshi and Reddy [43] developed a unified integro-differential nonlocal elasticity model and used this model to the bending of Euler–Bernoulli beams. Fernández-Sáez et al. [46] investigated the bending problems of Euler–Bernoulli beams using the Eringen integral constitutive equation. The closed-form bending solutions of Euler–Bernoulli beams and Timoshenko beams subjected to different loading and boundary conditions were carried out by Tuna and Kirca [16]. It appears that the first issue can be solved with the aid of the integro-differential nonlocal elasticity theory. The second issue is that one can only obtain a few natural frequencies of free vibrations of cantilever beams and that the counterintuitive stiffening effect is observed. This issue has been analytically solved by Xu et al. [47] using the weighted residual approaches (WRAs). In their work, they reformulated the variational-consistent boundary

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conditions, and presented the closed-form frequency solutions for Euler–Bernoulli beams and Timoshenko beams. The solutions of the above-mentioned issues demonstrate that, when one uses the nonlocal elasticity, the boundary conditions should be correctly employed rather than simply replacing the classical force resultants by the nonclassical force resultants in the equilibrium equations.

Within the framework of strain gradient elasticity theory, it emphasizes that materials constituting the body can be considered as atoms with higher-order deformation mechanism at small scales. Significant contributions in this field can be found in Mindlin and Tiersten [48], Toupin [49], recently in Yang et al. [50], Lam et al. [1] and Zhou et al. [51] and literature therein. This theory has been adopted to solve boundary value problems of static and dynamic behaviors of structures. For example, Papargyri-Beskou et al. [52] investigated the effects of the material length parameters on the bending and buckling of Euler–Bernoulli beams. Since then, the strain gradient theory has been widely used in modelling the micro- and nano-sized beams [53–67], plates [32,68–79] and shells [80–83]. These works show a stiffening effect for structures with characteristic sizes reducing to small sizes.

In order to bring both of the length scales into a combined elastic theory such that the stiffening effects and the softening effects of materials can be well described, Lim et al. [84] proposed a higher-order nonlocal strain gradient theory and applied the nonlocal strain gradient beam models to the study of the wave propagation. After that, several works dealing with buckling [85], free vibration [86,87] and wave propagation [88] of beams are reported. In these works, they emphasize that the classical results will be obtained for the same material length parameters. Since the partial differential order of the governing equation(s) of motion increases, the boundary value problems of structures modelled by the nonlocal strain gradient theory should be treated carefully. However, similar works have not been reported in the literature. For more details of nonlocal strain gradient models, one can refer to Papargyri-Beskou et al. [52], Li et al. [60], Akgöz and Civalek [57], Lazopoulos and Lazopoulos [59], Liang et al. [65] and Xu and Deng [66] for developing appropriate method to solve the boundary value problems.

The present paper is motivated by the fact that the higher-order boundary conditions induced by the nonlocal strain gradient theory should play a significant role on the buckling behaviors of Euler–Bernoulli beams. Therefore, the objective of the present work is to use the WRAs to derive the variational-consistent boundary conditions of nonlocal strain gradient beams, and to present the closed-form buckling solutions for beams subjected to various boundary conditions in which the effect of higher-order boundary conditions on the buckling loads is highlighted.

The structure of this paper is as follows. Section 2 briefly summarizes the nonlocal strain gradient theory. In Section 3, the governing equations of motion of nonlocal strain gradient Euler–Bernoulli beams in conjunction with the von Kármán nonlinear geometric relation are given, and the variational-consistent boundary conditions are derived by the WRAs. After the closed-form solutions of beam bending problems given in Section 4, the buckling solutions for beams subjected to three typical boundary conditions are addressed in Section 5. Finally, the conclusions are drawn in Section 6.

2. Nonlocal strain gradient theory

Motivated by the observations that materials at small scales exhibit either softening behaviors or stiffening behaviors, Lim et al. [84] developed an elastic theory which combines both the nonlocal elasticity theory and the strain gradient theory. Within the framework of this theory, the concept of the higher-order nonlocal strain gradient elasticity is proposed

$$\mathbf{t} = \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\sigma}^*, \tag{1}$$

where $\mathbf{t}$ is the total stress tensor of nonlocal strain gradient theory, ∇ is the gradient symbol. The stress tensor $\boldsymbol{\sigma}$ and higher-order stress tensor $\boldsymbol{\sigma}^*$ are given by

$$\boldsymbol{\sigma} = \int_V K_0(\mathbf{y}, \mathbf{x}, l_{e0}) \mathbf{C} : \boldsymbol{\varepsilon}^{\mathbf{y}}(\mathbf{y}) \, \mathrm{d}V \tag{2}$$

$$\boldsymbol{\sigma}^* = l_m^2 \int_V K_1(\mathbf{y}, \mathbf{x}, l_{e1}) \mathbf{C} : \nabla \boldsymbol{\varepsilon}^{\mathbf{y}}(\mathbf{y}) \, \mathrm{d}V \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the classical strain tensor, $\mathbf{C}$ is the usual fourth-order elasticity tensor, l_m is the internal length parameter, l_{e0}, l_{e1} are nonlocal parameters, $K_i(\mathbf{y}, \mathbf{x}, l_{ei}),\ i = 0, 1$ is the attenuation kernel function.

Mathematically, it is difficult to solve the above integral constitutive equations, Lim et al. [84] then, following the Eringen's method, introduced a simple constitutive equation

$$(1 - l_e^2 \nabla^2) \mathbf{t} = \mathbf{C} : \boldsymbol{\varepsilon} - l_m^2 \mathbf{C} : \nabla^2 \boldsymbol{\varepsilon} \tag{4}$$

For one-dimensional problems, the above constitutive equation reduces to

$$\left(1 - l_e^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) t_{xx} = E\varepsilon_{xx} - l_m^2 E\varepsilon_{xx,xx}. \tag{5}$$

Note that Eq. (5) contains two material length parameters. The first one indicates the nonlocal effect, and the second one denotes the size effect due to the higher-order strain gradient. Additionally, the nonlocal elasticity [8] and the strain gradient theory [89–91] can be obtained by taking $l_m = 0$ and $l_e = 0$, respectively.

3. Basic equations of nonlocal strain gradient beams

For preliminaries, we first present in Section 3.1 the main procedures developed in the literature to the boundary value problems of the nonlocal strain gradient beams. How the boundary conditions are obtained can be easily identified. Then, we use the WRAs to formulate the variational-consistent boundary conditions in Section 3.2.

3.1. Governing equations: A summary

We consider an elastic beam of length L, width b and thickness h. The x-axis is taken along the length of the beam, and z-axis is along the thickness of the beam. According to the Euler–Bernoulli beam theory, the displacements (u_1, u_2, u_3) along the (x, z) coordinate directions are given by

$$u_1(x,z) = u(x) - zw',\ u_2(x,z) = 0,\ u_3(x,z) = w(x), \tag{6}$$

where u, w are the axial and the transverse displacements of the beam mid-plane; the prime denotes the spatial differentiation with respect to variable x.

The only non-vanishing strain for a beam under large displacements can be captured by the von Kármán nonlinear strain, i.e.,

$$\varepsilon_{xx} = u_1' + \frac{1}{2} u_3'^2 = u' + \frac{1}{2} w'^2 - zw'', \tag{7}$$

where ε_{xx} is the longitudinal strain.

Next, we will present the detailed derivation of the governing equation and boundary conditions. With this aim at hand, we first give the following virtual work of the strain energy as follows