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Short Communication

A second order model for transient heat conduction in a slab with convective boundary conditions

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Abstract

In this paper we present a second order model for transient heat conduction in a slab obtained by using a perturbation method. We show that this simple model is accurate even for high values of the Biot number in a region surrounding the center of the slab.

Keywords: Heat conduction; Lumped model; Step response

1. Introduction

It is of interest for engineering calculations to propose simple mathematical formulations to deal with heat conduction problems. In the present work, we treat the unsteady heat conduction in a slab with convective boundary conditions. Typical applications may be found in the thermal study of building envelopes, metallurgical processes which involve heat treatment of steel strips and in many other industries. The well-established classical lumping procedure is extensively used [1–5]. However it is known to be inaccurate for great values of the Biot number which measures the competition between heat conduction in the slab and the convective heat exchanged with the surrounding fluid. Further it gives only the temperature of the surface of the considered medium as it has been shown in [6], where we have given a first order model for the slab, the infinite cylinder and the sphere. In the following sections, we extend the mathematical developments given in [6], to the second order, in the case of a slab. It will be shown that this leads to a simple and accurate model even for very high values of the Biot number in a region surrounding the center of the slab. The lower bounds of this region are given. Finally, a computational example for an infinite Biot number demonstrates the accuracy of the method.

2. Problem formulation

We consider unsteady heat conduction in a slab of thickness L, initially at a uniform temperature T_i. At $t = 0$, the solid material is exposed to a fluid at a variable temperature $f(t)$ with a heat transfer coefficient h. It is assumed that the thermophysical properties are constant. By using the dimensionless parameters defined by

$$\theta = \frac{T - T_i}{T_\infty - T_i} \quad X = \frac{x}{L} \quad Fo = \frac{\alpha t}{L^2} \quad B = \frac{hL}{\lambda}$$

where T_∞ is a reference temperature which will be the fluid constant temperature, the mathematical formulation of the problem is given by

$$\frac{\partial \theta}{\partial Fo} = \frac{\partial^2 \theta}{\partial X^2} \tag{1}$$

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$$\theta(X,0) = 0 \quad \text{at } t = 0 \tag{2}$$

$$-\frac{\partial\theta}{\partial X} = B(F-\theta) \quad \text{at } X = 0 \tag{3}$$

$$-\frac{\partial\theta}{\partial X} = B(\theta-F) \quad \text{at } X = 1 \tag{4}$$

In the previous equations, T is the temperature, t the time, x the spatial coordinate, α and λ are the thermal diffusivity and the conductivity of the solid material.

When the fluid temperature is constant ($f(t) = T_\infty$), the analytical solution [3] is given by

$$\theta(x,t) = 1 - 2\sum_{n=1}^{\infty} e^{-\mu_n^2\alpha t} \frac{\sin\left(\frac{\mu_n L}{2}\right)}{\frac{\mu_n L}{2} + \sin\left(\frac{\mu_n L}{2}\right)\cos\left(\frac{\mu_n L}{2}\right)}\cos(\mu_n x) \tag{5}$$

where μ_n are the positive roots of the following characteristic equation:

$$\tan\left(\frac{\mu_n L}{2}\right) = \frac{2B}{\mu_n L} \tag{6}$$

It is then necessary to calculate the eigenvalues for each value of the considered Biot number.

3. Perturbation solution

By introducing a small perturbation parameter in Eq. (1), the solution can be written in the form of an infinite series [6]:

$$\theta(X,\tau) = \sum_{n=0}^{\infty} \varepsilon^n \Psi_n(X) F_n(\tau) \tag{7}$$

By truncating the series, temperature can be expressed at order 1 and 2 as

$$\theta(X,\tau) = F(\tau) + \Psi_1(X)\frac{dF}{dFo} \tag{8}$$

$$\theta(X,\tau) = \Psi_0(X)F(Fo) + \Psi_1(X)\frac{dF}{dFo} + \Psi_2(X)\frac{d^2F}{dFo^2} \tag{9}$$

where

$$\Psi_0 = 1 \tag{10}$$

$$\Psi_1 = \frac{X^2}{2} - \frac{X}{2} - \frac{1}{2Bi} \tag{11}$$

$$\Psi_2(X) = \frac{X^4}{24} - \frac{X^3}{12} - \frac{X^2}{4B} + \frac{B+6}{24B}X + \frac{B+6}{24B} \tag{12}$$

Taking the Laplace transform of Eqs. (8) and (9), it follows:

$$\widehat{\theta}(X,p) = (1 + p\Psi_1(X))\widehat{F}(p) \tag{13}$$

$$\widehat{\theta}(X,p) = (1 + p\Psi_1(X) + p^2\Psi_2(X))\widehat{F}(p) \tag{14}$$

where $\widehat{\theta}$ and $\widehat{F}$ are the Laplace transforms of θ and F, respectively.

The following approximate transfer functions are obtained at order 1 and 2:

$$H_1^* = \frac{1}{1 - p\Psi_1} \tag{15}$$

$$H_2^* = \frac{1}{1 + p\Psi_1 + p^2\Psi_2} \tag{16}$$

Inverting back to the time domain these functions leads to the step responses:

$$\theta_1(\tau) = \int_0^\tau I_1(\tau)\,d\tau \tag{17}$$

$$\theta_2(\tau) = \int_0^\tau I_2(\tau)\,d\tau \tag{18}$$

where I_1 and I_2 are the impulse responses.

The previous developments allow us to calculate the temperature at any position of the solid domain when the temperature of the fluid is uniform at T_∞. By using Eqs. (15) and (17), the first order step response for the center temperature can be determined as follows:

$$\theta_{1C}(F) = \left(1 - \exp\left(-\frac{8B}{B+4}\cdot Fo\right)\right) \tag{19}$$

It has been shown in [6] that this first order solution is more accurate that the classical lumped model.

4. Second order step response

The second order transfer function at any position of the slab reads:

$$H_2 = 1 + \left(\frac{X^2}{2} - \frac{X}{2} - \frac{1}{2B}\right)p + \left(\frac{X^4}{24} - \frac{X^3}{12} - \frac{X^2}{4B} + \frac{B+6}{24B}X + \frac{B+6}{24B^2}\right)p^2 \tag{20}$$

One then obtains the following approximate transfer function:

$$H_2^* \approx \frac{1}{1 - \left(\frac{X^2}{2} - \frac{X}{2} - \frac{1}{2B}\right)p + \left(\frac{5X^4}{24} - \frac{5X^3}{12} + \frac{X^2}{4B}(B-1) + \frac{6-B}{24B}X - \frac{1}{24B}\right)p^2} \tag{21}$$

The first pole of this function is always negative (even for an infinite Biot number). The second pole however, is positive beyond some values of X for all the Biot numbers as shown in Fig. 1, where we present this pole as a function of X for $B = 1$ and 100 000. Although one can calculate the region where the pole is positive for each value of B, we only give here the limits of this region for an infinite Biot number. In this case one can show that the two poles are negative if:

$$\frac{1}{2} - \frac{\sqrt{5}}{10} < X < \frac{1}{2} + \frac{\sqrt{5}}{10} \tag{22}$$