European Journal of Mechanics B/Fluids 59 (2016) 52–56

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Symmetry in the problem of wave modes of thin viscous liquid layer flow

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ARTICLE INFO

Article history:
Received 2 June 2015
Received in revised form
18 March 2016
Accepted 16 April 2016
Available online 30 April 2016

Keywords:
Falling liquid films
Coordinate transformation
Symmetry
Nonlinear waves

ABSTRACT

The equations in conservative form for modelling nonlinear waves on a liquid film flowing down a vertical plane have been investigated. It has been found out that the equations with boundary conditions are invariant under parity transformation in the extended computational domain. The steady-state travelling solutions are numerically shown to have the detected symmetry for moderate Reynolds numbers. The use of this symmetry for the numerical solution of the problem by Galerkin methods significantly increases the efficiency of calculations.

1. Introduction

Film flows are widely used in heat and mass transfer technologies. Theoretical study of the problem started in 1916 with Nusselt's work that has become fundamental for the theory of film condensation. The research provided the exact solution for the waveless flow of a thin layer of viscous liquid over an inclined solid wall. However, the inherent development of waves on free film surfaces further intensifies heat transfer from the heated substrate. The study of the basic characteristics and the form of these waves in the pioneer works of Kapitza [1] laid the foundation for a new scientific direction.

Many of the researchers (see, for example, [2,3]) use low-dimensional Galerkin methods to derive the reduced systems of equations for coefficients at respective basis functions, depending on the transverse coordinate. With a lucky choice of the basis such approaches result in rather small number of equations (two or three). In our work we reveal a specific symmetry of governing equations and a wide class of their solutions, including those of interest from experimental point of view. The awareness about this symmetry allows two-fold reduction of the number of equations and gives a better idea of the success of the known models, derived earlier by extrapolation of analytical results obtained only for small Reynolds numbers.

The full problem formulation for waves on the isothermal flowing film is clear and includes a system of Navier–Stokes and continuity equations with appropriate boundary conditions on the wall and at the free surface. This formulation implies problem solving in a changing flow area unknown in advance, which greatly complicates the mathematical and numerical simulation.

One of the approaches to solving the moving boundary problem appeared in the mid 80s of the last century. It comes to rewriting of hydrodynamic equations in new variables, transforming the flow area into the strip of constant thickness (Fig. 1):

$$x = x, \qquad \eta = \frac{y}{h(x,t)}, \quad t = t. \tag{1}$$

Here h is an instant local film thickness. The coordinate system (1) is non-orthogonal, so the normal vector formulation of equations is inapplicable. For this reason, many authors (see for example [4,5]) reduce the method to a simple change of variables without transformation of vectors and tensors, contained in the original equations.

For the case of a free falling film on a vertical plane, the transformation (1) was performed in the hydrodynamic equations, written in tensor form invariant under arbitrary coordinate transformation in [6]. As a result, the following system was obtained for the case of moderate liquid flow rates in the long wavelength approximation, i.e. when characteristic length of longitudinal perturbations is much larger than the film thickness:

$$\frac{\partial(hu)}{\partial t} + \frac{\partial(hu^2)}{\partial x} + \frac{\partial(huv)}{\partial \eta} = \frac{\sigma}{\rho} h \frac{\partial^3 h}{\partial x^3} + \frac{\nu}{h}\frac{\partial^2 u}{\partial \eta^2} + gh \tag{2}$$

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http://dx.doi.org/10.1016/j.euromechflu.2016.04.004
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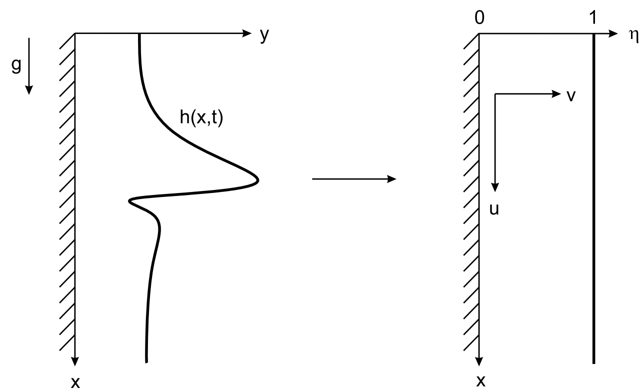

Fig. 1. The flow area in x–y and x–η coordinates.

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}+\frac{\partial (hv)}{\partial \eta}=0. \tag{3}$$

Here σ is the surface tension, ρ is the density, ν is the kinematic viscosity, g is the free fall acceleration, u and v are contravariant components of velocity corresponding to the coordinates x and η, respectively. It is clear that since x remains an original Cartesian coordinate, the component $u(x,\eta,t)$ remains the longitudinal component of physical velocity u_c in contrast to $v(x,\eta,t)$, related to the Cartesian component of the transverse velocity v_c as follows:

$$v_c=\eta\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\right)+hv.$$

Boundary conditions on the solid wall are:

$$u(x,0,t)=0,\qquad v(x,0,t)=0. \tag{4}$$

No-stress and kinematic conditions on the free surface are:

$$\frac{\partial u}{\partial \eta}(x,1,t)=0 \tag{5}$$

$$v(x,1,t)=0. \tag{6}$$

There are three unknown functions in this problem, and in the meantime, h is the function of only two variables (x,t). In the calculated domain (x,η) there are 2 unknown functions—$u(x,\eta,t)$ and $v(x,\eta,t)$, therefore, 2 equations are necessary to find them. In the long-wave model these two equations are: the continuity equation and the momentum balance in the longitudinal direction. A kinematic condition on the free surface ($\eta=1$) allows finding the function $h(x,t)$. In contrast to the regular presentation of the kinematic condition on a free boundary in Cartesian coordinates, where it is represented as a differential equation:

$$v_c(x,h,t)=\frac{\partial h}{\partial t}+u_c(x,h,t)\frac{\partial h}{\partial x},$$

it takes the form (6) in curvilinear coordinates (1).

2. Symmetry

Let new transverse coordinate be:

$$\eta'=\eta-1,$$

then the flow area lies in the interval $\eta'\in[-1,0]$. Note that η' is expressed in the original Cartesian coordinates as follows:

$$\eta'=\frac{y}{h(x,t)}-1.$$

It is easy to see that Eqs. (2)–(3) are invariant under the transformation:

$$\eta'\to-\eta' \tag{7}$$

$$u(x,\eta',t)\to u(x,-\eta',t) \tag{8}$$

$$v(x,\eta',t)\to -v(x,-\eta',t). \tag{9}$$

It means in particular that there are two types of solutions of these equations in the extended strip $\eta'\in[-1,1]$. Solutions of the first type are characterized by symmetry:

$$u(x,\eta',t)=u(x,-\eta',t) \tag{10}$$

$$v(x,\eta',t)=-v(x,-\eta',t). \tag{11}$$

It is clear that the solutions of the first type satisfying the no-slip and no-penetration boundary conditions on both boundaries:

$$u(x,-1,t)=v(x,-1,t)=0 \tag{12}$$

$$u(x,1,t)=v(x,1,t)=0 \tag{13}$$

are the solutions to original problem (2)–(6) in a half-strip $[-1,0]$. Indeed on the boundary $\eta'=-1$, the conditions (4) are satisfied, and the boundary condition of no-penetration at $\eta'=0$ (kinematic condition $v(x,0,t)=0$) is met automatically, due to fact that the contravariant transverse component of the velocity v for the first type solutions is an odd function with respect to η'. And since the function u is even, the dynamic condition on this boundary is automatically fulfilled as well:

$$\frac{\partial u}{\partial \eta'}(x,0,t)=0.$$

Solutions of the second type do not have such symmetry, but in virtue of (7)–(9), if a solution $u_1(x,\eta',t)$ and $v_1(x,\eta',t)$ exists, then there is a solution:

$$u_2(x,\eta',t)=u_1(x,-\eta',t)$$

$$v_2(x,\eta',t)=-v_1(x,-\eta',t).$$

At that

$$u_1(x,\eta',t)\neq u_1(x,-\eta',t)$$

$$v_1(x,\eta',t)\neq -v_1(x,-\eta',t).$$

However, if the second type solutions for the problem (2)–(6) are extended into the interval $\eta'\in[0,1]$ then, the boundary conditions (13) at $\eta'=1$ are not necessarily fulfilled.

Considering higher orders of smallness makes the problem as a whole and boundary conditions on free surface (5) in particular much more complicated. This will most probably lead to breaking of symmetry (7)–(9). Anyway, the mentioned symmetry will be destroyed even at the used approximation if other conditions at free boundary, e.g. accounting for gas influence on film flowing, are considered. Nevertheless, the found symmetries for the boundary layer model are quite interesting per se. In particular, the above property of the symmetry is useful for finding solutions to the problem (2)–(6).

3. Calculations

For numerical solution the problem (2)–(6) was written in a dimensionless form:

$$\varepsilon Re\left(\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{h}\right)+\frac{\partial}{\partial \eta'}\left(\frac{QV}{h}\right)\right)=\frac{1}{h^2}\frac{\partial^2 Q}{\partial \eta'^2}+3h+\frac{3^{1/3}Fi^{1/3}}{Re^{5/3}}\varepsilon^3 Reh\frac{\partial^3 h}{\partial x^3} \tag{14}$$

$$\frac{\partial h}{\partial t}+\frac{\partial Q}{\partial x}+\frac{\partial V}{\partial \eta'}=0 \tag{15}$$

$$Q(x,-1,t)=V(x,-1,t)=0 \tag{16}$$

$$\frac{\partial Q}{\partial \eta'}(x,0,t)=0,\qquad V(x,0,t)=0. \tag{17}$$

Here, we introduce new functions $Q=uh$, $V=vh$ relative to which Eq. (3) is linear, $Re=u_0h_0/\nu$ is the Reynolds number, $Fi=$