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Weakly nonlinear convective regimes in a horizontal fluid layer with poorly conducting boundaries and temperature-dependent viscosity

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ABSTRACT

In this work, the influence of temperature dependence of viscosity on the weakly nonlinear regimes in a horizontal fluid layer with poorly conducting boundaries is studied. A multi-scale method is used to derive the amplitude equations, which are investigated analytically and numerically. The stability of roll patterns, square cells, hexagonal cells and quasi-periodic dodecagonal structures is investigated analytically. As a numerical method we use one of modifications of the spectral method. It has been found that only hexagonal and square patterns maintain stability, the hexagonal pattern being always stable for sufficiently large values of the Prandtl number. The calculations have shown that there is the region of co-existence of hexagonal and square patterns for a small range of the parameter, which is responsible for temperature dependence of viscosity. The hexagonal cells are always excited subcritically, whereas the square cells can be excited both in a subcritical and supercritical manner. Our investigations at small values of the Prandtl number have also revealed the instability, which is associated with nonzero vertical vorticity.

1. Introduction

In the literature, there have been a sufficiently large number of papers, which investigate the influence of temperature dependence of viscosity on flows of different fluids and convection. These problems are of great interest in lubrication, tribology, food processing, instrumentation, viscometry and in the field of the study of convection in the Earth's atmosphere, the Earth's hydrosphere and the Earth's mantle (and in mantles and atmospheres of some other planets).

The convection in layers with poorly conducting boundaries has also been studied extensively. In particular this problem is of interest due to the study of convection in the Earth's mantle.

At first we consider the papers investigating the stability of non-convectional flows. The stability of Couette flow has been studied in the articles [1–7]. The stability of channel and pipe flows is considered in the papers [8–17]. The effect of viscous heating is included in some of these studies. A mixed Rayleigh–Benard–Poiseuille convection has been studied in [18].

Now we consider the papers investigating the influence of temperature-dependent viscosity on convection. Extensive studies have been undertaken to investigate the influence of temperature dependence of viscosity on convection in a horizontal fluid layer, the boundaries of which have perfect thermal conductivity. The first experimental study on this problem has been made by Graham [19]. He studied convection in air. After this problem was experimentally investigated by Tippelskirch [20,21]. In the first paper he studied convection in liquid sulfur. In the second paper he studied convection in aerosols. These authors observed convection in the form of hexagonal cells. The first theoretical studies on weak temperature dependence of viscosity have been made by Palm [22], Segel and Stuart [23], Palm and Oiann[24] and Busse [25]. They have found out that convection in this case appears in the form of hexagonal cells. Busse [25] has also shown that when the Rayleigh number is slightly higher than the critical value, a hexagonal pattern is replaced by rolls.

The problem of convection in a horizontal layer with fixed thermal flux on the rigid boundaries was first investigated by Nepomnyashchy [26]. He derived the amplitude equation, found analytically the stationary solution and analytically investigated the stability of this solution. The problem of convection in a horizontal layer with poorly conducting boundaries was first investigated by Busse and Riahi [27]. They used the method of

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expansion in a small parameter. Somewhat later, Chapman and Proctor [28] independently from Nepomnyashchy explored the case of fixed thermal flux on boundaries. The cases of rigid, stress-free and mixed boundaries were considered. They derived the amplitude equations, found analytically the stationary solutions and investigated these equations numerically. As an extension of their study, Proctor [29] using the same approach derived the amplitude equations for the problem of Busse and Riahi [27]. To explore these equations he used a variational method. Jenkins [30] investigated the case of finite boundary conductivity. Jenkins [31] considered the temperature dependence of viscosity for the case of finite boundary conductivity, and briefly reviewed the case of poorly conducting boundaries. For poorly conducting boundaries he wrote the amplitude equations, the coefficients of which after second nondimensionalization are comparable with the coefficients of the amplitude equations with the finite but small conductivity. However, he restricted himself to the case of square cells and roll pattern. It should also be noted that like many other authors he confined his investigation to the case of the infinite Prandtl number. In some of the above mentioned articles it is stated that their weakly nonlinear results can be extended to the finite Prandtl case. However, this statement is true only in part. When the problem is solved for small temperature perturbations this statement turns to be reasonable at least for rigid boundaries. But if the temperature perturbations are finite (with the Rayleigh number deviation remaining small as before, and the wave length being large), this is not the case. This fact was first mentioned by Pismen [32]. The same inference follows from the paper by Lyubimov and Cherepanov [33], who investigated the case of inhomogeneous heating in the layer with fixed thermal flux at the boundaries. In this case, we get an additional equation, which describes the vertical vorticity.

In this study, we considered Jenkin's problem but in the case of the finite Prandtl number. The results of this study are also of interest for the infinite Prandtl number since Jenkin's investigation ignores structures that are more complicated than the square cells and roll patterns.

2. Formulation of the problem

We consider a horizontal fluid layer of depth d located between two rigid plates of depth λd. The fluid has density ρ, thermal conductivity κ, specific heat capacity c_p. The plates have density $\tilde{\rho}$, thermal conductivity $\tilde{\kappa}$ and specific heat capacity $\tilde{c}_p$. It is assumed that the thermal conductivity of slabs $\tilde{\kappa}$ is much less than the thermal conductivity of fluid κ. Also it is assumed that the fluid density ρ and dynamic viscosity of the fluid linearly depend on the temperature:

$$\rho = \rho_*(1 - \beta(T - T_*)), \tag{1}$$

$$\eta = \eta_*(1 - \psi(T - T_*)), \tag{2}$$

where T_* is the temperature in the center of the layer in the absence of convection. Usually people consider more complex temperature dependence of viscosity (for example, exponential). But we suggest weak dependence of viscosity on temperature (the coefficient ψ is small). In this case we can expand any function into Taylor series on $(T - T_*)$. The first term of expansion will be linear (in some special cases this term can be zero, but we does not consider such cases).

The plates are assumed to obey the Fourier law:

$$\tilde{\rho}\tilde{c}_p \frac{\partial \tilde{T}}{\partial t} = \tilde{\kappa}\nabla^2 \tilde{T}. \tag{3}$$

In the fluid layer the Boussinesq approximation is used:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho_*}\nabla p + \frac{1}{\rho_*}\nabla\cdot\mathbf{S} + g\beta(T - T_*)\mathbf{e_z}, \tag{4}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{5}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right) = \kappa\nabla^2 T, \tag{6}$$

where $\mathbf{S}$ is the viscous stress tensor, which is defined by

$$S_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{7}$$

In this case the following boundary conditions are satisfied:

$$z = -d(1+\lambda):\ T = T_2, \tag{8}$$

$$z = d(1+\lambda):\ T = T_1, \tag{9}$$

$$z = \pm d:\ T = \tilde{T}, \qquad \kappa\frac{\partial T}{\partial z} = \tilde{\kappa}\frac{\partial \tilde{T}}{\partial z}. \tag{10}$$

The system of equations and boundary conditions admits the solution that corresponds to the mechanical equilibrium of the fluid:

$$\mathbf{v_0} = 0, \tag{11}$$

$$T_0 = T_* - qz, \tag{12}$$

$$z \geq d:\ \tilde{T}_0 = T_* - qd(\zeta - 1)/\zeta - qz/\zeta, \tag{13}$$

$$z \leq -d:\ \tilde{T}_0 = T_* + qd(\zeta - 1)/\zeta - qz/\zeta, \tag{14}$$

where $\zeta = \tilde{\kappa}/\kappa$, and $q = (T_2 - T_1)\zeta/(2d(\lambda + \zeta))$ is the vertical thermal flux through the fluid in the absence of the fluid motion. This solution is also called the basic state. In this study, we considered the case of small ζ, which means that the quantity q can be expanded into series in ζ:

$$q = q_0(1 - \zeta/\lambda + \cdots), \tag{15}$$

where the following notation is used:

$$q_0 \equiv (T_2 - T_1)\zeta/(2d\lambda). \tag{16}$$

Here q_0 is the main part of the temperature gradient in the fluid, which is considered to be finite. This implies that $T_2 - T_1$ is large and has the order of magnitude of the quantity $1/\zeta$.

After introducing the perturbations of the basic state, we adopt for pressure renormalization the following scales: d is the length scale, $d^2\rho_* c_p/\kappa$ is the time scale, $\kappa/(\rho_* c_p d)$ is the velocity scale, $q_0 d$ is the temperature scale (both for the fluid layer and the plates), $\eta_*\kappa/(d^2\rho_* c_p)$ is the scale of pressure and viscous stress tensor. As a result, Eqs. (3)–(6) and boundary conditions (8)–(10) take the form (we use the same notations for dimensional and nondimensional forms):

$$\alpha\frac{\partial \tilde{\vartheta}}{\partial t} = \zeta\nabla^2\tilde{\vartheta}, \tag{17}$$

$$\frac{1}{Pr}\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbf{S} + Ra\,\vartheta\,\mathbf{e_z}, \tag{18}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{19}$$

$$\frac{\partial \vartheta}{\partial t} + \mathbf{v}\cdot\nabla\vartheta = \nabla^2\vartheta + \frac{q}{q_0}v_z, \tag{20}$$

$$z = \pm(1+\lambda):\ \tilde{\vartheta} = 0, \tag{21}$$

$$z = \pm 1:\ \vartheta = \tilde{\vartheta},\ \frac{\partial \vartheta}{\partial z} = \zeta\frac{\partial \tilde{\vartheta}}{\partial z}, \tag{22}$$