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Frequency downshift in a viscous fluid

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ABSTRACT

In this paper, we derive a viscous generalization of the Dysthe (1979) system from the weakly viscous generalization of the Euler equations introduced by Dias et al. (2008). This "viscous Dysthe" system models the evolution of a weakly viscous, nearly monochromatic wave train on deep water. It includes only one free parameter, which can be determined empirically. It contains a term that provides a mechanism for frequency downshifting in the absence of wind and wave breaking. The system does not preserve the spectral mean. Numerical simulations demonstrate that the spectral mean typically decreases and that the spectral peak decreases for certain initial conditions. The linear stability analysis of the plane-wave solutions of the viscous Dysthe system demonstrates that waves with frequencies closer to zero decay more slowly than waves with frequencies further from zero. Comparisons between experimental data and numerical simulations of the nonlinear Schrödinger, dissipative nonlinear Schrödinger, Dysthe, and viscous Dysthe systems establish that the viscous Dysthe system accurately models data from experiments in which frequency downshifting was observed *and* experiments in which frequency downshift was not observed.

1. Introduction

In the late 1970s, Lake et al. [1] and Lake and Yuen [2] conducted physical experiments that investigated the evolution of nonlinear wave trains on deep water. They found that the growth of the Benjamin–Feir instability is followed by a shift in the spectral peak, the frequency corresponding to the Fourier mode with largest magnitude, to a frequency closer to zero. Subsequent experiments, including those in Su et al. [3] and Melville [4], demonstrated that the amplitude of the lower sideband grows and eventually overtakes that of the carrier wave. These later experimental studies focused on waves with larger steepness and involved wave breaking. More recently, Segur et al. [5] conducted similar experiments without wave breaking or wind. They found that frequency downshifting (FD) is not observed (in their tank) if the waves have what they refer to as "small or moderate" amplitudes and that FD is observed if the amplitude of the carrier wave is "large" or if the sideband perturbations are "large enough". They also found that if FD occurs then (i) the spectral mean (defined below) decreases monotonically as the waves travel down the tank and (ii) FD occurs in the higher harmonics before it occurs in the fundamental. The goal of the current work is to provide a mathematical justification for FD that does not rely on wind or wave breaking.

There are two quantities that are commonly used to quantify frequency downshifting: the spectral peak, ω_p, and the spectral mean, ω_m. The spectral mean (in Hertz) at a location x in tank is given by

$$\omega_m(x) = \frac{\mathcal{P}(x)}{\mathcal{M}(x)}, \tag{1}$$

where $\mathcal{M}$ (in cm^2) and $\mathcal{P}$ (in cm^2/s) are defined by

$$\mathcal{M}(x) = \frac{1}{L}\int_0^L |B|^2 dt, \tag{2a}$$

$$\mathcal{P}(x) = \frac{i}{2L}\int_0^L \left(BB_t^* - B_t B^*\right) dt. \tag{2b}$$

Here B (in cm) is a measure of the slow evolution of the complex envelope of a nearly monochromatic train of plane waves; x (in cm) is the horizontal coordinate in the direction of wave propagation; t (in s) is the temporal coordinate; L (in s) is the t period of B; and * represents complex conjugate. We note that while some studies use $\mathcal{P}$ by itself as a measure of FD, we focus on ω_p and ω_m.

Zakharov [6] derived the cubic nonlinear Schrödinger (NLS) equation from the Euler equations as a model for the evolution of the envelope of a nearly monochromatic wave group. The NLS

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equation preserves the spectral mean, so it cannot be used to model FD. Dysthe [7] carried out the NLS perturbation analysis one order higher to obtain what is now known as the Dysthe system. Lo and Mei [8] numerically solved the NLS equation and Dysthe system and established that the Dysthe system more accurately predicts the evolution of mildly sloped, narrow-banded, weakly nonlinear waves over longer time periods than does the NLS equation. They also found that dissipative generalizations of the Dysthe system are required to model waves of moderate steepness over long distances. Finally, their numerical studies showed that the Dysthe system did not lead to a permanent FD even though the Dysthe system does not preserve the spectral mean. Segur et al. [5] established that the dissipative NLS equation accurately models the evolution of wave trains in which no FD occurred and that it cannot model FD because it preserves the spectral mean. The dissipative NLS equation was used as an ad-hoc model without formal justification until Dias, Dyachenko, and Zakharov [9] derived it from a weakly viscous generalization of the Euler equations. The first step in the current work is to carry out the Dysthe perturbation analysis starting from the Dias, Dyachenko, and Zakharov weakly viscous generalization of the Euler equations in order to derive a new system, which we call the viscous Dysthe system. Readers interested in wind and wave-breaking justifications for FD are referred to Trulsen and Dysthe [10], Hara and Mei [11], Kato and Oikawa [12], Brunetti and Kasparian [13], Brunetti et al. [14], and Islas and Schober [15].

The paper is organized as follows: Section 2 contains the derivation of the viscous Dysthe system. Section 3 contains a summary of the properties of this new equation. Section 4 contains comparisons of viscous Dysthe predictions with experimental data from two experiments, one of which exhibited FD.

2. Derivation of the viscous Dysthe equation

Dias, Dyachenko, and Zakharov [9] introduced the following system for an infinitely-deep, weakly viscous fluid

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0, \quad \text{for } -\infty<z<\eta, \tag{3a}$$

$$\phi_t+\frac{1}{2}\left|\nabla\phi\right|^2+g\eta=-2\bar{\nu}\phi_{zz}, \quad \text{at } z=\eta, \tag{3b}$$

$$\eta_t+\eta_x\phi_x+\eta_y\phi_y=\phi_z+2\bar{\nu}\Delta\eta, \quad \text{at } z=\eta, \tag{3c}$$

$$|\nabla\phi|\to 0, \quad \text{as } z\to-\infty. \tag{3d}$$

Here $\phi=\phi(x,y,z,t)$ represents the velocity potential of the fluid; $\eta=\eta(x,y,t)$ represents the free-surface displacement; t is the temporal coordinate; x and y are horizontal coordinates; z is the vertical coordinate; g represents the acceleration due to gravity; and $\bar{\nu}>0$ represents the kinematic viscosity of the fluid. This model assumes that gravity and viscosity are the only forces acting on the fluid. The Euler equations are obtained from (3) by setting $\bar{\nu}=0$.

As frequency downshifting is most commonly observed in time series recorded at various locations in the direction of wave propagation, assume

$$\begin{aligned}\eta(x,y,t) &= \epsilon^3\bar{\eta}+\epsilon B e^{i\omega_0 t-ik_0x}\\ &\quad+\epsilon^2B_2e^{2(i\omega_0t-ik_0x)}+\cdots+c.c.,\end{aligned} \tag{4a}$$

$$\begin{aligned}\phi(x,y,z,t) &= \epsilon^2\bar{\phi}+\epsilon A_1e^{k_0z+i\omega_0t-ik_0x}\\ &\quad+\epsilon^2A_2e^{2(k_0z+i\omega_0t-ik_0x)}+\cdots+c.c.,\end{aligned} \tag{4b}$$

where $\omega_0>0$ and $k_0>0$ represent the frequency and wave number of the carrier wave respectively; $\epsilon=2a_0k_0\ll 1$ is a dimensionless parameter known as the wave steepness; a_0 represents a typical wave amplitude; and $c.c.$ stands for complex conjugate. Assuming $k_0>0$ determines the z dependence in ϕ. The choice $\omega_0>0$ implies that the wave train travels in the positive x direction as t increases. The A's and $\bar{\phi}$ depend on the slow variables $X=\epsilon x$, $Y=\epsilon Y$, $Z=\epsilon z$, and $T=\epsilon t$, while the B's and $\bar{\eta}$ depend on X, Y and T. Next, assume

$$A_j=A_{j0}+\epsilon A_{j1}+\epsilon^2A_{j2}+\epsilon^3A_{j3}+\cdots,\quad \text{for } j=1,2,3,\ldots, \tag{5a}$$

$$B_j=B_{j0}+\epsilon B_{j1}+\epsilon^2B_{j2}+\epsilon^3B_{j3}+\cdots,\quad \text{for } j=2,3,4,\ldots, \tag{5b}$$

$$\bar{\eta}=\bar{\eta}_0+\epsilon\bar{\eta}_1+\epsilon^2\bar{\eta}_2+\cdots, \tag{5c}$$

$$\bar{\phi}=\bar{\phi}_0+\epsilon\bar{\phi}_1+\epsilon^2\bar{\phi}_2+\cdots. \tag{5d}$$

Following the work of Dias, Dyachenko, and Zakharov, assume that viscous effects are small by setting $\bar{\nu}=\epsilon^2\nu$. Substituting (4) and (5) into (3) and completing the fourth-order perturbation analysis gives the deep-water dispersion relationship,

$$gk_0=\omega_0^2, \tag{6}$$

the system that defines B and $\bar{\phi}_0$,

$$\begin{aligned}&2i\omega_0\Big(B_T+\frac{g}{2\omega_0}B_X\Big)\\ &+\epsilon\Big(\frac{g}{4k_0}B_{XX}-\frac{g}{2k_0}B_{YY}+4gk_0^3|B|^2B+4ik_0^2\omega_0\nu B\Big)\\ &+\epsilon^2\Big(-\frac{ig}{8k_0^2}B_{XXX}+\frac{3ig}{4k_0^2}B_{XYY}+2igk_0^2B^2B_X^*\\ &+12igk_0^2|B|^2B_x+2k_0\omega_0B\bar{\phi}_{0X}-8k_0\omega_0\nu B_X\Big)=0,\\ &\text{at } Z=0,\end{aligned} \tag{7a}$$

$$\bar{\phi}_{0Z}=2\omega_0\Big(|B|^2\Big)_X, \quad \text{at } Z=0, \tag{7b}$$

$$\bar{\phi}_{0XX}+\bar{\phi}_{0ZZ}=0, \quad \text{for } -\infty<Z<0, \tag{7c}$$

$$\bar{\phi}_{0Z}\to 0, \quad \text{as } Z\to-\infty, \tag{7d}$$

and

$$B_2=k_0B^2+\mathcal{O}(\epsilon), \tag{8a}$$

$$B_3=\frac{3}{2}k_0^2B^3+\mathcal{O}(\epsilon). \tag{8b}$$

Eqs. (8a) and (8b) define the leading-order contributions to the amplitudes of the second and third harmonics of the carrier wave respectively.

Interchanging the forms of the assumptions in Eqs. (5a) and (5b) and following a similar procedure while focusing on the leading-order term of the velocity potential instead of the leading-order term of the surface displacement gives

$$\begin{aligned}&2i\omega_0\Big(A_T+\frac{g}{2\omega_0}A_X\Big)\\ &+\epsilon\Big(\frac{g}{4k_0}A_{XX}-\frac{g}{2k_0}A_{YY}+4k_0^4|A|^2A+4ik_0^2\omega_0\nu A\Big)\\ &+\epsilon^2\Big(-\frac{ig}{8k_0^2}A_{XXX}+\frac{3ig}{4k_0^2}A_{XYY}-2ik_0^3A^2A_X^*\\ &+12ik_0^3|A|^2A_x+2k_0\omega_0A\bar{\phi}_{0X}-8k_0\omega_0\nu A_X\Big)=0,\\ &\text{at } Z=0,\end{aligned} \tag{9a}$$

$$\bar{\phi}_{0Z}=\frac{2k_0^2}{\omega_0}\Big(|A|^2\Big)_X, \quad \text{at } Z=0, \tag{9b}$$

$$\bar{\phi}_{0XX}+\bar{\phi}_{0ZZ}=0, \quad \text{for } -\infty<Z<0, \tag{9c}$$

$$\bar{\phi}_{0Z}\to 0, \quad \text{as } Z\to-\infty. \tag{9d}$$

In essence, this system defines the evolution of the variable A_1 in Eq. (5a).