European Journal of Mechanics B/Fluids 50 (2015) 9–18

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Dispersion relations for gravity water flows with two rotational layers

Calin Iulian Martin

Institut für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

ARTICLE INFO

Article history:
Received 12 December 2013
Received in revised form
25 October 2014
Accepted 27 October 2014
Available online 7 November 2014

Keywords:
Dispersion relation
Discontinuous vorticity
Stability of the laminar solutions

ABSTRACT

We derive the dispersion relation for periodic traveling water waves propagating at the surface of water possessing a layer of constant non-zero vorticity γ_1 adjacent to the free surface above another rotational layer of vorticity γ_2 which is adjacent to the flat bed. As a by-product we give necessary and sufficient condition for local bifurcation in the frame-work of piecewise constant vorticity. Moreover, we give estimates on the speed at the free surface of the bifurcating laminar flows. These estimates involve only the vorticity γ_1, the mean depth of water d and the depth at which the jump in vorticity occurs. A stability result for certain laminar flows is also presented.

1. Introduction

We devote this paper to the subject of wave–current interactions [1–3] which despite its recognized importance has seen little advancement—a circumstance generated by the complexity of the problem. The term "current" describes here a flow with a flat free surface. The prevailing feature of currents is the existence of shear in the vertical direction. The extensive studies by Peregrine [4] and Jonsson [2] document the interaction of surface gravity waves with vertically sheared currents.

Even the uniform currents (i.e. irrotational flows), which are the simplest ones, have awaited a long time until they had a firm theoretical basis that came through the extensive studies of the Stokes waves [5] and the flow beneath them concerning particle trajectories, behavior of the pressure [6–9]. The substantial progress in the more complicated scenario of a non-uniform current came only relatively recently through [10] where the existence of small and large amplitude steady periodic water waves with a general (regular) vorticity distribution was proved. Paper [10] was followed by a bulk of papers treating a variety of topics like symmetry [11–13], stability [14], regularity of the free surface and of the stream lines [15–19] and allowing for more sophisticated features like stratifications [20–22], stagnation points and critical layers [23–27] or the presence of a singular (merely bounded or piecewise constant) vorticity distribution [28–31].

As far as our paper is concerned we shall deal here with non-uniform currents whose main characteristic is the presence of non-zero vorticity in the flow and, in addition, we assume that the vorticity has a discontinuous piecewise constant distribution. This situation is of practical relevance and can be observed in regions where there is a rapid change of the current strength cf. [2]. The distribution of vorticity in our setting is as follows: we consider a layer of constant non-vanishing vorticity γ_1 adjacent to the free surface above a rotational flow of vorticity γ_2. On physical grounds, this situation is justified by the fact that rotational wind generated waves possess a layer of high vorticity adjacent to the wave surface [32,33], while in the near bed region there may exist currents resulting from sediment transport along the ocean bed [34].

The main topic that we address here is the dispersion relation for small-amplitude waves. This relation indicates how the relative speed of the bifurcating laminar flow at the free surface varies with respect to certain parameters like the wave-length, the mean depth of the flow, and – in the case of a piece-wise constant vorticity like the one we consider here – the position of vorticity jumps. The dispersion relation we obtain recovers the corresponding formula (23) from [35] for the case of a layer of constant non vanishing vorticity adjacent to the flat bed within an irrotational flow as well as the dispersion relation (81) from [28] in the context of a layer of constant non-zero vorticity adjacent to the free surface above fluid in irrotational flow.

To treat the above mentioned vorticity distribution we adopt the framework of weak solutions to the free boundary Euler equations and we refer the interested reader to [28] where the existence of steady two-dimensional periodic water waves of small and large amplitude in a flow with an arbitrary bounded (but discontinuous) vorticity was proven in the context of a fixed mass flux. For the context of a fixed mean-depth we refer the reader to [36]. Concerning the main topic of our paper, it was shown in [37,38] that the dispersion relations corresponding to the fixed mean depth approach

E-mail address: calin.martin@univie.ac.at.

http://dx.doi.org/10.1016/j.euromechflu.2014.10.005
0997-7546/

coincide with those in [28,35] corresponding to the fixed mass flux point of view.

We also like to mention that the dispersion relation for capillary–gravity waves for the situation of a layer of vorticity adjacent to the surface above irrotational fluid as well as for the case of an isolated layer of vorticity adjacent to the flat bed was obtained in [39]. For recent results on dispersion relations for small amplitude gravity waves with continuous non-constant vorticity we refer the reader to [40].

We briefly summarize the content of the paper. After introducing the equations of motion in Section 2, we derive in Section 3 the dispersion relation. Here we also give necessary and sufficient conditions which ensure that certain laminar flows give rise to waves of small-amplitude. In Section 4 we present a stability result for laminar flows in the spirit of [14].

2. The equations of motion

This paper considers two-dimensional steady periodic water waves which travel over a rotational, incompressible and inviscid fluid propagating in the positive x-direction over the flat bed $y = -d$ (for some $d > 0$) with the free surface $y = \eta(x)$ being a small perturbation of the flat free surface $y = 0$. We assume that the only restoring force acting upon the fluid is gravity. In a reference frame moving in the same direction as the wave with wave speed $c > 0$, the equations of motion are Euler's equations

$$\begin{cases} (\mathfrak{u} - c)\mathfrak{u}_x + \mathfrak{v}\mathfrak{u}_y = -P_x \\ (\mathfrak{u} - c)\mathfrak{v}_x + \mathfrak{v}\mathfrak{v}_y = -P_y - g, \end{cases} \tag{2.1}$$

together with the incompressibility condition

$$\mathfrak{u}_x + \mathfrak{v}_y = 0, \tag{2.2}$$

whereby $(\mathfrak{u}, \mathfrak{v})$ denotes the velocity field, P is the pressure and g is the gravitational constant. An assumption that we make throughout the paper is that $(\mathfrak{u}, \mathfrak{v})$, P and the surface wave profile $x \to \eta(x)$ are periodic in the variable x and for simplicity we choose the period $L = 2\pi$. The vorticity (assumed to be piecewise constant) of the flow is

$$\omega := \mathfrak{u}_y - \mathfrak{v}_x.$$

Eqs. (2.1) and (2.2) are supplemented by the kinematic boundary conditions

$$\begin{cases} \mathfrak{v} = (\mathfrak{u} - c)\eta_x \quad \text{on } y = \eta(x) \\ \mathfrak{v} = 0 \quad \text{on } y = -d \end{cases} \tag{2.3}$$

representing essentially a necessary and sufficient condition for the flow to move along a boundary but not across/through the boundary, and the dynamic boundary condition

$$P = P_{atm} \quad \text{on } y = \eta(x), \tag{2.4}$$

which decouples the motion of the air above the free surface from that of the water. Here P_{atm} denotes the constant atmospheric pressure. The details about the validity of (2.1)–(2.4) are worked out in [1]. To simplify the problem just presented we introduce the stream function ψ defined (up to a constant) by the relations

$$\psi_x = -\mathfrak{v}, \qquad \psi_y = \mathfrak{u} - c.$$

One reasonable assumption (true for waves which are not near breaking) is the absence of stagnation points in the flow. This assumption can be analytically written as

$$\mathfrak{u} < c \quad \text{throughout the fluid.} \tag{2.5}$$

Due to (2.5) we have cf. [1,10] that the vorticity ω is a single-valued function of ψ, i.e.,

$$\omega(x, y) = \gamma(-\psi(x, y)),$$

which finally yields the reformulation of (2.1)–(2.4) as the free boundary value problem

$$\begin{cases} \Delta\psi = \gamma(-\psi) & \text{in } -d < y < \eta(x), \\ |\nabla\psi|^2 + 2g(y + d) = Q & \text{on } y = \eta(x), \\ \psi = 0 & \text{on } y = \eta(x), \\ \psi = -p_0 & \text{on } y = -d, \end{cases} \tag{2.6}$$

where Q is a constant related to the total head, and $p_0 < 0$ is a constant representing the relative mass flux, given by

$$p_0 = \int_{-d}^{\eta(x)} (\mathfrak{u}(x, y) - c)\, dy.$$

We aim to further simplify the problem (2.6) by transforming it into a problem in the fixed domain $\overline{\Omega} := [-\pi, \pi] \times [p_0, 0]$. The latter task is performed by means of the partial hodograph transform

$$q(x, y) = x, \qquad p(x, y) = -\psi(x, y) \tag{2.7}$$

which, due to assumption (2.5), provides a diffeomorphism from the fluid domain to Ω and renders the problem (2.6) into the quasilinear elliptic boundary value problem

$$\begin{cases} (1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma h_p^3 = 0 & \text{in } \overline{\Omega}, \\ 1 + h_q^2 + (2gh - Q)h_p^2 = 0 & \text{on } p = 0, \\ h = 0 & \text{on } p = p_0, \end{cases} \tag{2.8}$$

where the unknown function h defined on $\overline{\Omega}$ by

$$h(q, p) := y + d$$

represents the height above the flat bed and is even and of period 2π in the q-variable. The absence of stagnation points is now equivalent to the elliptic non-degeneracy condition

$$h_p > 0 \quad \text{in } \overline{\Omega}.$$

The discontinuous vorticity regime requires a weak formulation of the above system as done in [28] where the authors showed that (2.8) is equivalent to the problem

$$\begin{cases} \left\{ \dfrac{1 + h_q^2}{2h_p^2} + \Gamma(p) \right\}_p - \left\{ \dfrac{h_q}{h_p} \right\}_q = 0 & \text{in } \Omega, \\ \dfrac{1 + h_q^2}{2h_p^2} + gh = \dfrac{Q}{2} & \text{on } p = 0, \\ h = 0 & \text{on } p = 0, \end{cases} \tag{2.9}$$

whereby Γ is defined by

$$\Gamma(p) = \int_0^p \gamma(s)\, ds, \quad p \in [p_0, 0].$$

By a solution of (2.9) we understand a function $h \in W^{2,r}_{\text{per}}(\Omega) \subset C^{1,\alpha}_{\text{per}}(\overline{\Omega})$, with $r > \frac{2}{1-\alpha}$, (for a fixed $\alpha \in (1/3, 1)$) that is a generalized solution cf. [41, Section 8]. A family of laminar solutions, i.e. parallel shear flows with flat free surfaces, parametrized by $\lambda > 2 \max_{[p_0,0]} \Gamma$ is given by

$$\boldsymbol{h}(p) := \boldsymbol{h}(p, \lambda) = \int_0^p \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}} + \frac{Q - \lambda}{2g} \in C^{1,\alpha}([p_0, 0]) \tag{2.10}$$

cf. [28]. The parameter λ is related to wave speed at the flat free surface $y = 0$ of the laminar flow by the formula

$$\sqrt{\lambda} = (c - \mathfrak{u})|_{y=0} = \frac{1}{\boldsymbol{h}_p(0)},$$