European Journal of Mechanics B/Fluids 50 (2015) 38–51

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Equilibrium states of class-I Bragg resonant wave system

Dali Xu [a], Zhiliang Lin [a,*], Shijun Liao [a,b,*]

[a] State Key Laboratory of Ocean Engineering, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China

[b] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

HIGHLIGHTS

- The class-I Bragg resonant waves are solved analytically.
- Multiple equilibrium-state resonant wave systems with time-independent wave spectrum are found.
- Bifurcations with respect to wave propagation angle, water depth, bottom slope and nonlinearity are found.

ARTICLE INFO

Article history:
Received 6 April 2014
Received in revised form
30 October 2014
Accepted 30 October 2014
Available online 17 November 2014

Keywords:
Equilibrium state
Bragg resonance
Water waves
Homotopy Analysis Method (HAM)

ABSTRACT

In this paper, the class-I Bragg resonant waves are investigated in the case that a primary surface wave propagates obliquely over the bottom with ripples distributed in a very large area. Two kinds of equilibrium-state resonant wave systems with time-independent wave spectrum are found. In all cases, the primary and resonant wave components contain most of the wave energies. For the first kind, the primary and resonant wave components have the same amplitude. However, for the second kind, they contain different wave energies. Especially, the bifurcations of the equilibrium-state resonant waves with respect to the wave propagation angle, the water depth, bottom slope and nonlinearity are found for the first time. To the best of our knowledge, these two kinds of equilibrium-state class-I Bragg resonant waves and especially the bifurcations have never been reported. All of these might deepen and enrich our understanding about the Bragg wave resonance. Mathematically, unlike previous analytic approaches which regard the considered problem as an initial-value one, we search for the unknown equilibrium-state resonant waves from the viewpoint of boundary-value problem, using an analytic method that has nothing to do with small/large physical parameters at all.

1. Introduction

In his pioneering work, Phillips [1] found the resonance criterion of a quartet of progressive waves in deep water:

$$\boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 \pm \boldsymbol{k}_4 = 0, \qquad \omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 = 0, \tag{1}$$

where $\boldsymbol{k}_i$ denotes the wave number, $\omega_i = \sqrt{gk_i}$ with $k_i = |\boldsymbol{k}_i|$ being the angular frequency given by the linear wave theory in deep water, g is the acceleration due to gravity, respectively. Phillips [1] found that the amplitude of the resonant wave component, if it is zero initially, grows linearly with time. When Phillips' resonance criterion (1) is fully or nearly satisfied, Benney [2] established the evolution equations of wave mode amplitudes, and demonstrated the well-known time-dependent periodic exchanges of wave energy governed by Jacobian elliptic functions.

Same as the Stokes wave [3–5], there were some attempts to obtain the equilibrium states of this quartet resonance with the *time-independent* wave amplitude, angular frequency and wavenumber. In the context of perturbation techniques, the equilibrium states of the quartet resonance have *not* been found at an order higher than three, because perturbation results (mostly at the third order of approximation) contain the secular terms when Phillips' criterion is satisfied so that "the perturbation theory breaks down due to singularities in the transfer functions", as currently pointed out by Madsen and Fuhrman [6]. However, using an analytic approximation method for highly nonlinear problems, namely the homotopy analysis method (HAM) [7–13], Liao [14] successfully gained, for the first time, the equilibrium states of the quartet resonant progressive waves in deep water, which have no exchange of wave energy at all between different wave components. In addition, Liao [14] found that there exist multiple equilibrium states

* Corresponding authors.
E-mail addresses: linzhiliang@sjtu.edu.cn (Z. Lin), sjliao@sjtu.edu.cn (S. Liao).

http://dx.doi.org/10.1016/j.euromechflu.2014.10.006
0997-7546/

of this quartet resonance in deep water, and especially the resonant wave component may contain much *less* wave energy than the primary ones.

Based on the homotopy, a basic concept of topology, the HAM has many advantages over other analytic approximation techniques for nonlinear problems. First, unlike perturbation techniques, it has nothing to do with small/large physical parameters, and thus is valid for more problems in science and engineering. Besides, it provides us great freedom to choose the base-function and equation-type of equations for high-order approximations. Especially, different from all other analytic approximation methods, the HAM provides us a simple, convenient way to guarantee the convergence of solution series. In addition, the HAM has been proved to logically include some traditional analytic approximation methods, such as "the Lyapunov artificial small parameter method" proposed by the famous Russian mathematician Aleksandr Mikhailovich Lyapunov (1857–1918), "the Adomian decomposition method" which was developed from the 1970s to the 1990s by George Adomian, the chair of the Center for Applied Mathematics at the University of Georgia, USA, and so on. Thus, the HAM has rather general meanings in theory. The HAM has been successfully applied to solve many nonlinear problems in fluid mechanics, applied mathematics, physics, finance and so on. Especially, some new solutions were found by means of the HAM, which had never been reported and neglected by other analytic approximation methods and even by numerical techniques. All of these illustrate the validity and novelty of the HAM. For details about the HAM, please refer to the two books of Liao [8,11]. It should be emphasized that the multiple equilibrium states of resonant wave systems in deep water were first discovered by Liao [14] using the HAM.

In 2012, Xu et al. [15] further applied the HAM to solve this quartet resonance of progressive waves in *finite* water depth d with flat bottom, when Phillips' resonance criterion (1) with $\omega_i = \sqrt{gk_i \tanh k_i d}$ is exactly satisfied. They confirmed that the multiple equilibrium states of resonant waves also exist in finite water depth. Meanwhile, the resonant wave component might contain a small proportion of wave energy, too. Besides, they verified all of their conclusions using the Zakharov equation. In addition, Liu et al. [16] verify that the multiple equilibrium states also exist in the resonance of multiple waves. Current, Liu et al. [17] confirmed the existence of the steady-state resonant waves by experiments: their experimental results agree quite well with the theoretical ones reported in [16]. All of these confirmed the generality of the multiple equilibrium-states of resonant wave systems in deep and finite water.

A simple pendulum without damping, as plotted in Fig. 1, is a good analogy for the equilibrium-states of resonant wave systems. When the pendulum is at the lowest position initially, it will stay at this equilibrium position forever. When the pendulum is disturbed away from this equilibrium position, it oscillates periodically around it, with the periodic exchange of its potential energy and kinetic energy. Thus, equilibrium-states of a dynamic system are fundamental and important for us to have a global understanding of it. Some complicated dynamic systems have multiple equilibrium positions. The resonant waves as dynamical systems are much more complicated than a simple pendulum: they have multiple equilibrium states. The equilibrium states of the resonant wave systems found by Liao [14] in deep water and Xu et al. [15] in finite water depth are like the equilibrium positions of a complicated dynamical system. Such kind of equilibrium states determine the global characteristics of the dynamic system and thus belong to a kind of fundamental property. Therefore, it is very important to determine these equilibrium states of resonant waves, which are helpful to deepen and enrich our understandings about resonant waves. Note that such kind of equilibrium states are rather special

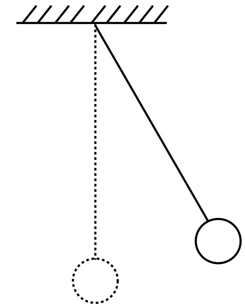

Fig. 1. Equilibrium state of dynamical systems.

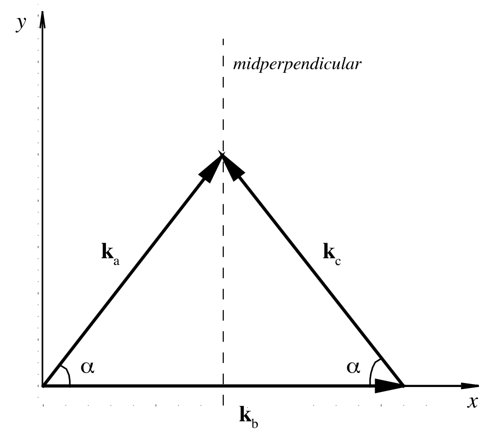

Fig. 2. Sketch map of the class-I Bragg resonant waves.

in practice. In most cases, there often exist the time-dependent periodic exchanges of wave energy around these equilibrium states, which can be described by the evolution equations of wave mode amplitudes given by Benney [2]. However, Benney [2] did not report the existence of the multiple equilibrium states of resonant waves, which were first found by Liao [14] in deep water, confirmed by Xu et al. [15] in water of finite depth and Liu et al. [16] for multiple wave interactions by means of the HAM.

Therefore, the equilibrium states of resonant waves are important in physics for a better understanding of global characteristic of a given resonant wave system. Do multiple equilibrium states of resonant wave systems exist generally in other more complicated cases? The answer is positive, as revealed in this article.

It is well-known that the resonance occurs for nonlinear wave–bottom interaction, too. The simplest case is known as the class-I Bragg resonance. It occurs when a primary surface wave propagates over an undulated bed that contains ripples with a single wavenumber $\boldsymbol{k}_B$. Without loss of generality, let $\boldsymbol{k}_A$ denote the wavenumber of the primary wave and $\boldsymbol{k}_C$ that of the resonant one, respectively. Note that the names of the so-called primary and resonant waves can be interchanged, since there exists a kind of symmetry on the perpendicular bisector of the bottom wavenumber $\boldsymbol{k}_B$, as shown in Fig. 2 and illustrated later. So, without loss of generality, we can simply call them wave A and wave C, too.

The corresponding class-I Bragg resonance criterion reads

$$\boldsymbol{k}_A - \boldsymbol{k}_B = \boldsymbol{k}_C, \qquad \omega_A = \omega_C, \tag{2}$$

where

$$\omega_A = \sqrt{g|\boldsymbol{k}_A| \tanh |\boldsymbol{k}_A| d}, \qquad \omega_C = \sqrt{g|\boldsymbol{k}_C| \tanh |\boldsymbol{k}_C| d}$$

denote the wave angular frequency in the linear theory and d is the water depth, respectively. The resonant wave results from the resonant interaction between the primary wave and the bottom.