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Overturned internal capillary–gravity waves

Benjamin F. Akers [a,*], David M. Ambrose [b], Kevin Pond [a], J. Douglas Wright [b]

[a] Department of Mathematics and Statistics, Air Force Institute of Technology, Dayton, OH, United States

[b] Department of Mathematics, Drexel University, Philadelphia, PA, United States

HIGHLIGHTS

- Global bifurcation branches of internal capillary–gravity waves are computed.
- All possible behaviors from a global bifurcation theorem are realized.
- A numerical method is developed for computing the boundary of bifurcation surfaces.
- The role of the waves' second harmonic in its bifurcation structure is discussed.
- Steep waves limited by self-intersection at both crests and troughs are computed.

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ABSTRACT

A vortex sheet formulation of irrotational, incompressible Euler flow is used to compute periodic traveling waves at the interface between two constant-density, two-dimensional fluids, including waves with overturned crests. Branches of traveling waves are computed via numerical continuation, which are jointly continuous in the physical parameters: Bond number, Atwood number and mean shear. Global branches are computed, for various choices of parameters, illustrating the termination criteria of the global bifurcation theorem of Ambrose et al. (2015). The dependence of the branches, and their termini, on the physical parameters are probed via a boundary continuation method. Bifurcation surfaces are computed; these surfaces are both overturned and self-intersecting. The connection between the second harmonic of a Stokes' wave expansion and the shape of these surfaces is discussed.

1. Introduction

We study the irrotational, incompressible Euler equations at the interface between two constant-density fluids; these are an upper fluid and a lower fluid. The fluid regions are infinitely deep in the vertical direction and periodic in the horizontal direction. We seek traveling wave solutions, or solutions for which the free surface is of permanent form and steadily translating. Waves are computed on this interface numerically, including the effects of the physical parameters of surface tension, gravity, mean shear, and density ratio. We compute large amplitude solutions, including those with overturned crests or troughs, up to the limit of self-intersection.

Since the fluids are irrotational in the bulk, the vorticity is equal to zero inside either fluid region. The velocity may jump at the interface (specifically, the tangential component of the velocity may jump, while the normal component must be continuous) [1], thus, the vorticity is not identically zero but is instead measure-valued and supported only on the interface. The interface is thus referred to as a vortex sheet.

We denote the densities of the fluids as ρ_1 and ρ_2, which can each be any non-negative, constant value (not both zero). A useful non-dimensional quantity, then, is the Atwood number, $\mathrm{At} = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$. The surface tension parameter is τ, which is taken to be a positive constant, and the constant acceleration due to gravity is g, which may be any real value.

The present work has its foundation in prior work by three of the authors. In [2], a novel formulation for the interfacial traveling wave problem was introduced, and was used for both analysis and computing. In particular, in the case in which the two fluids have equal density (i.e., $\mathrm{At} = 0$), a local bifurcation theorem was applied to show the existence of small-amplitude traveling waves, for any value of the mean shear, for fixed, nonzero surface tension. Numerical solutions were computed using a quasi-Newton method in Fourier space, similar to [3]. Subsequently, the same authors followed up in [4], in which water waves were studied. The water wave is the special case of the vortex sheet in which the upper fluid is taken to have density equal to zero (i.e., $\mathrm{At} = 1$); the

* Corresponding author.
E-mail address: benjamin.akers@afit.edu (B.F. Akers).

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gravity parameter was able to be taken to be positive, negative, or zero. The analysis of this work used the implicit function theorem to demonstrate that Crapper waves [5], which are a family of exact, pure capillary traveling water waves, can be perturbed through the inclusion of gravity; see also [6,7] for further developments in this area. The computational portion of [4] again used a quasi-Newton method in Fourier space to compute these gravity-perturbed Crapper waves; a new wave of maximum amplitude was found when the gravity parameter takes a specific small, negative value.

Further analysis has since been carried out by two of the authors and Strauss [8]. In this work, a global bifurcation theorem was proved, for traveling Stokes' waves between fluids of arbitrary constant densities. In the case that the two fluids have different densities, the main theorem of [8] specializes to the following:

Theorem 1. *For all choices of the surface tension parameter $\tau > 0$, the spatial periodicity parameter $M > 0$, the mean shear parameter $\gamma_0 \in \mathbb{R}$, the densities of the fluids $\rho_1, \rho_2 \geq 0$ (with $\rho_1 \neq \rho_2$) and the gravity parameter $g \in \mathbb{R}$, there exists a countable number of connected sets of smooth non-trivial symmetric periodic traveling wave solutions (bifurcating from a quiescent equilibrium) for the two-dimensional gravity–capillary vortex sheet problem. Each of these connected sets has at least one of the following properties:*

(a) *it contains waves whose interfaces have lengths which are arbitrarily long;*
(b) *it contains waves whose interfaces have curvature which is arbitrarily large;*
(c) *it contains waves where the jump in the tangential component of the fluid velocity across the interface or its derivative is arbitrarily large;*
(d) *its closure contains a wave whose interface contains a point of self intersection;*
(e) *it contains a sequence of waves whose interfaces converge to a flat configuration but whose speeds contain at least two convergent subsequences whose limits differ.*

One might say that a shortcoming of the theory of global bifurcations is that, while a variety of possible behaviors along bifurcation curves can be identified, the theory does not generally identify which of these behaviors in fact occur. We thus address this question via simulation. We have been able to find all of the behaviors, (a) through (e), computationally, for some choices of parameter values. For example, cases (a), (b), and (c) all occur in the density matched cases, and are reported in [2]. Case (d) occurs at the generic choice of parameter values in this work, and is well known to occur for the Crapper family of waves ($\mathrm{At} = 1, g = 0$) [5]. The most controversial is case (e), since in analytical work in the absence of surface tension, this phenomenon can typically be ruled out by a maximum principle argument; one example of such an argument is in [9]. In the presence of surface tension, the maximum principle argument is not available because of the larger number of derivatives. We find that outcome (e) can occur for certain negative values of the gravity parameter; this is illustrated in Fig. 1.

In addition to computing individual branches of traveling waves, we seek to understand how these branches depend on the physical parameters. We focus on the termini of these branches, seeking to observe how the extreme wave's character varies from branch to branch. The extreme wave in the generic case is self intersecting, case (d) of the global bifurcation theorem. We ask whether this extreme wave, with a self-intersecting profile, includes a bubble (or droplet) entrained into the upper or lower fluid. We explore the extent to which small amplitude asymptotics can be used to predict this behavior. We also compute surfaces on which traveling waves exist (global bifurcation branches with one of the physical parameter varied) and observe the character of the boundaries of these surfaces. We develop a new numerical method to compute these boundaries called BCM (boundary continuation method). We observe that these surfaces, just like the waves, are both overturned and self-intersecting. BCM allows us to compute traveling waves which are not on a global bifurcation branch as described by Theorem 1, i.e. they are not on branches of traveling waves which are connected to small amplitude with *fixed* physical parameters.

There are numerous studies of the similar problem without surface tension. For example, the case of $\tau = 0$ with At varying has been numerically studied by Vanden-Broeck and Turner [10,11]. They observe that the maximal waves are near turning points in the speed–amplitude plane (i.e., the bifurcation curves spiral in). Another early study is by Meiron and Saffman with $\tau = 0$ and different values of At.

With surface tension, the special cases $\mathrm{At} \in \{0, 1\}$ have been studied several times, including, as we have mentioned, by some of the authors in [2,4,8]. Modern studies of the case $\mathrm{At} = 1$, with small surface tension $\tau \approx 0$, include both the infinite [12] and finite depth cases [13]. A host of classical and overturned traveling water waves ($\mathrm{At} = 1$) have been computed by Okamoto and colleagues, many of which are presented in the manuscript [14]. Our numerical work differs in flavor from much of that in the literature in our focus on the global bifurcation picture, computing the location in parameter space of waves of extremal displacement and self intersection. The numerical methods described herein require that the vortex sheet does not self intersect; traveling waves can be computed with self intersections (or bubbles) via other methods, see [15].

Other works have considered the case of two fluids, an upper layer of finite depth and a vacuum above; in this setting, there is an interface between the two fluids, and an upper free surface [16,17]. Finally, we mention that there are also experimental works on this subject, such as [18–21]. Of course, in addition to periodic traveling waves, solitary waves are also considered, for example in the numerical work [22].

The works just described all considered the Euler equations. Of course, in interfacial fluid dynamics, there are also many approximate models which have been developed, such as the Korteweg–de Vries equation and the Benjamin–Ono equation, among others. Some relevant papers using model equations are [23–26]. Another kind of approximation technique is amplitude expansion methods; the papers [27–30] make such expansions in the manner of Stokes. The beginnings of such an amplitude expansion for internal waves on the vortex sheet are derived herein.

The remainder of the paper is organized as follows. In Section 2, we will give the equations of motion for our capillary–gravity interfacial fluid problem. This includes giving the traveling wave ansatz, as developed by the authors in [2]. In Section 3, a weakly nonlinear theory is developed, and the second harmonic of a Stokes' wave is calculated for generic parameter values. In Section 4, our numerical methods are described, including descriptions of two methods for exploring parameter space: one based on adaptive sampling and another which uses continuation to trace the boundaries of where traveling waves exist (BCM). Numerical results from these algorithms are given in Section 5. Conclusions and future research areas are presented in Section 6.

2. Formulation

We start from the formulation developed by the same authors in [2,4], which we now describe. The traveling wave equations are derived from the evolution equations for a vortex sheet at the interface of two incompressible irrotational fluids. As with any free boundary problem, both the interface and its evolution must be described. We write the free surface as $(x(\alpha, t), y(\alpha, t))$, define