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Unified derivation of thin-layer reduced models for shallow free-surface gravity flows of viscous fluids

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ABSTRACT

We propose a unified framework to derive thin-layer reduced models for some shallow free-surface flows driven by gravity. It applies to incompressible homogeneous fluids whose momentum evolves according to Navier–Stokes equations, with stress satisfying a rheology of viscous type (i.e. the standard Newtonian law with a constant viscosity, but also non-Newtonian laws generalized to purely viscous fluids and to viscoelastic fluids as well). For a given rheology, we derive various thin-layer reduced models for flows on a rugous topography slowly varying around an inclined plane. This is achieved thanks to a coherent simplification procedure, which is formal but based on a mathematically clear consistency requirement between scaling assumptions and the approximation errors in the differential equations. The various thin-layer reduced models are obtained depending on flow regime assumptions (either fast/inertial or slow/viscous). As far as we know, it is the first time that the various thin-layer reduced models investigated here are derived within the same mathematical framework. Furthermore, we obtain new reduced models in the case of viscoelastic non-Newtonian fluids, which extends Bouchut and Boyaval (2013).

1. Introduction

The flow models built with Navier–Stokes equations for viscous fluids have been simplified in various ways for a long time. This has resulted in a large number of reduced models, in particular, numerous *thin-layer models* for *shallow free-surface flows* often obtained by a formal asymptotic analysis [1–4].

Initially, reduced models were looked after because they were more amenable to analytical computations than full models. For instance, the Stoker and Ritter solutions to the inviscid Saint-Venant (i.e. shallow-water) equations have allowed one to model dam breaks in a simple way, with analytical formulas. Nowadays, computer simulations often yield good approximations to full models. But good simulations of complex full models are expensive (time-consuming at least), and typically less easily interpreted from the physical viewpoint. In the case physical parameters of the model have to be explored, reduced versions of the model may thus still be preferred to full models, for instance to discriminate against various possible rheologies by comparison with experiments, see e.g. [5,6]. Moreover, in the case where *many values* of the physical parameters have to be explored, reduced models (computationally cheaper than full models) often remain the single numerical option (even for simple toy-models, computational reductions can prove crucial, see e.g. the case of stochastic parameters in [7]). Reduced models thus remain very useful. But it is also desirable to compare them with full models, or at least one another (in the case of varying physical parameters). Now, rigorous error bounds between various (full or reduced) models are available in simple cases only, where the models remain of the same kind (see e.g. [7]). The case of free-surface shallow flow models for fluids driven by gravity is particularly striking, since various thin-layer reduced models have been proposed (see the numerous references later in this work), which are of different mathematical type, depending on various assumptions about the flow regime considered during their derivation (i.e. depending on solution properties that are not obviously connected to data, and assumed instead).

The case of free-surface flow models for perfect fluids driven by gravity (non-necessarily-shallow flows of inviscid fluids) has been treated recently: a unifying approach to irrotational water-wave models could be constructed recently [8] and extended to

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new reduced models with vorticity [9]. For shallow free-surface flows (of non-necessarily perfect fluids), a generic procedure has also been used recently to derive thin-layer models with various rheologies [1,4], but it seems to hold only for the flow regimes that we later term "slow", and it has not been used for all the cases treated in the present work (viscoelastic fluids for instance).

Our primary goal here is to establish a mathematical framework where various thin-layer reduced models obtained in various flow regimes (slow or fast), given a fixed possibly viscous rheology, can be connected one another. Moreover, we treat various rheologies (Newtonian and non-Newtonian) of viscous (also viscoelastic) fluids in the various regimes. We believe that we have thereby unified, for the first time, the derivation of many various thin-layer reduced models for shallow free-surface gravity flows, for Newtonian and non-Newtonian (viscous or viscoelastic) fluids in slow and fast regimes.

Our mathematical simplification procedure is formal. It cannot certify *rigorously* that a solution to the reduced model is a good approximation of a solution to the full model. But it is based on an intuitive *coherence* property with a clear mathematical formulation: the consistency between scaling assumptions and approximation errors in the equations. Moreover, given one rheology, we invoke successive assumptions about the flow regime until the simplification procedure delivers a closed reduced model that is coherent with the original full model. In a given flow regime, for one given rheology, our procedure is thus univoque.

The simplification procedure is inspired by [2,3] where the viscous shallow water equations are derived from the Navier–Stokes equations for Newtonian (purely viscous) fluids (see Section 3). It aims at building a consistent approximation to a family of solutions to the initial full model, when the family of solutions defines an adequate asymptotic regime for shallow free-surface flows of incompressible viscous fluids driven by gravity on a rugous topography. Consistency is required asymptotically with respect to a nondimensional parameter $\varepsilon > 0$ parameterizing the solutions.

The asymptotic regime is defined such that only *long* free-surface waves are captured when $\varepsilon \to 0$ (i.e. only piecewise constants). The asymptotic regime in turn constrains the topographies that one can consider at the bottom of an *incompressible* flow of a *homogeneous* fluid. Precisely, in the present work, we consider only topographies defined by *slow variations around a flat plane* inclined by a constant angle θ with respect to the gravity field, thus asymptotically long waves too. Extensions with asymptotically long variations of θ seem possible [10] but are not considered here, for the sake of simplicity.

Invoking the Navier–Stokes momentum balance equations (as opposed to Euler equations), with a viscous dissipative term in the bulk along with friction boundary conditions of Navier-type on the rugous bottom of the flow, is crucial to the reduction procedure developed here (compare with [11,12]). This modelling choice motivates the assumption (H4): $\partial_z \boldsymbol{u}_H = O(1)$ on the shear rate, a key step to derive *coherent* reduced models (see e.g. (33)). It is of course the responsibility of modellers to check if it makes sense for application to a real shallow flow (see Remark 1). In any case, the asymptotics $\varepsilon \to 0$ is an *idealization*. In practice, one should ask if solutions of the reduced model are close to solutions of the initial model, i.e. if they can be corrected for $\varepsilon > 0$ small and give physically-interesting answers: this justifies our *coherence* requirement.

Finally, one obtains here a synthetic view of various existing simplifications of the Navier–Stokes equations, for various rheologies and various flow regimes. Moreover, new reduced models for fluids with complex rheologies are derived.

- For viscous Newtonian fluids (modelled by the standard Navier–Stokes equations), we obtain either viscous shallow water equations in fast (inertial) flow regimes (as e.g. in [2,3]) or lubrication equations in slow (viscous) flow regimes (as e.g. in [13,1,14]), see Section 4.
- For viscous non-Newtonian fluids (nonlinear power-law models), we obtain either a nonlinear version of the shallow water equations in fast flow regimes that is apparently new, or nonlinear lubrication equations in slow flow regimes (see [4] and references therein), see Section 5.
- For viscoelastic non-Newtonian fluids, we obtain either shallow water equations with additional stress terms which extends the recent work [11] in fast flow regimes, or new lubrication equations in slow flow regimes (different than those in [15,16]), see Section 6.

A few remarks are also in order.

- The case of perfect fluids (no internal stresses) is singular. We recover it here as the inviscid limit of viscous models, provided friction is small enough at the rugous bottom boundary, and it yields the same reduced model whatever the explicit formulation of the viscous terms. One obtains the thin-layer model of Saint-Venant [17] widely used in hydraulics, also known as the nonlinear shallow-water equations. A dissipative term associated with the Navier friction boundary condition remains. It could have been derived more straightforwardly without viscosity, like in [12] for viscoelastic fluids with zero retardation time, but then also less naturally (conditions tangent to the boundary are not required for perfect fluids).
- The case of viscoplastic Non-Newtonian fluids (i.e. Bingham-type fluids) occurs as a singular limit of the nonlinear power-law models. This case is very interesting from the modelling viewpoint (some fluids are believed to possess a yield-stress, which suits well for modelling fluid–solid transitions like e.g. in avalanches). But it is also difficult from the mathematical viewpoint (the model is undetermined below the yield-stress) as well as from the physical viewpoint (the yield-stress concept is still debated [18]).
- In the case of viscoelastic fluids, we improve here the model derived in [11]. Note that the constitutive equations that we use here are simple and alike in [11]. (They are linear equations in the conformation tensor state variable, physically-consistent from the frame-invariance and from the molecular theories viewpoint.) However, here, we additionally take into account: friction at the bottom, an inclination between the constant gravity field and the main direction of the flow, surface tension, two-dimensional effects and a purely Newtonian additional viscosity (equivalently, a non-zero retardation time, from the viscoelastic rheology viewpoint).

For a physically-inclined review of thin-layer models in many possible flow regimes, we recommend [14], and the older one [13] with a focus on stability.

2. Mathematical setting of the problem

We endow the space $\mathbb{R}^3$ with a Galilean reference frame using Cartesian coordinates $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$. We denote by a_x (respectively a_y, a_z) the component in direction $\boldsymbol{e}_x$ (resp. $\boldsymbol{e}_y$, $\boldsymbol{e}_z$) of a vector (that is a rank-1 tensor) $\boldsymbol{a}$, by $a_{xx}, a_{xz}, \ldots$ the components of higher-rank tensors, by $\boldsymbol{a}_H$ the vector of "horizontal" components (a_x, a_y), by $(\boldsymbol{a}_H)^\perp = (-a_y, a_x)$ an orthogonal vector, by $\nabla_H a$ the horizontal gradient $(\partial_x a, \partial_y a)$ of a smooth function $a : (x, y) \to a(x, y)$, and by $D_t a$ the material time-derivative $\partial_t a + (\boldsymbol{u} \cdot \nabla) a$. We use the Frobenius norm $|\boldsymbol{a}| = \mathrm{tr}(\boldsymbol{a}^T \boldsymbol{a})^{1/2}$ for tensors.

We consider gravity flows of incompressible homogeneous fluids, which are governed by Navier–Stokes equations (momentum balance and mass continuity)

$$\begin{cases} D_t \boldsymbol{u} = \mathrm{div}(\boldsymbol{S}) + \boldsymbol{f} & \text{in } \mathcal{D}(t), \\ \mathrm{div} \boldsymbol{u} = 0 & \text{in } \mathcal{D}(t), \end{cases} \tag{1}$$