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Instabilities of the sidewall boundary layer in a rapidly rotating split cylinder

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ABSTRACT

The instabilities of the sidewall boundary layer in a rapidly rotating split cylinder are studied numerically. Axisymmetric results are studied extensively where a variety of different states are found. In the basic state, the interior flow is in solid-body rotation with the mean rotation rate of the two cylinder halves. The sidewall boundary layer of the basic state is compared with theoretical results. For sufficiently fast mean rotation and large enough differential rotation between the two halves, instabilities in the boundary layer appear. These instabilities result in periodic and quasi-periodic states in different parameter regimes. The instabilities are localized in the boundary layer, but they may transport shear into the interior if their associated frequencies are less than twice the mean rotation frequency, and then only in the form of inertial wave beams along directions determined by their frequencies.

1. Introduction

The structure of the sidewall boundary layer in a rapidly rotating cylinder subjected to some differential rotation has attracted much attention because of both its practical and fundamental importance. Stewartson [1] first showed that when the sidewall rotates at a rate slightly faster than the two endwalls, the sidewall boundary layer has a sandwich structure consisting of an inner layer whose thickness scales as $Re^{-1/3}$ (where Re is the rotation Reynolds number based on the mean angular velocity of the cylinder, its radius and the kinematic viscosity of the fluid) and an outer layer with a thickness that scales as $Re^{-1/4}$. The $Re^{-1/4}$ layer is where the perturbation to the azimuthal velocity is adjusted and the inner $Re^{-1/3}$ layer is needed to adjust the secondary meridional flow. The boundary layers on the endwalls are of Ekman type, and they scale as $Re^{-1/2}$.

Hocking [2] considered another differentially rotating cylinder flow consisting of a split cylinder with one half rotating slightly faster than the other. His analysis considered the case of an infinitely long cylinder in which case the meridional flow in the sidewall layer due to the flows being pumped out of the Ekman layers at the ends was neglected. The finite split cylinder problem, in which endwall effects are present, was later addressed by van Heijst [3] using boundary layer analysis. His results showed that the quasi-geostrophic (almost independent of the axial direction) $Re^{-1/4}$ layer is unable to provide the matching between the azimuthal interior velocity and the discontinuous velocity of the sidewall, that the non-geostrophic $Re^{-1/3}$ layer is needed to match the discontinuous sidewall velocity but is unable to do so on its own, but that the combination of the two layers does provide the required matching.

The theoretical boundary layer analysis proceeds in the limit of very fast rotation (large Re) and very small differential rotation. The small differential rotation allows one to neglect the inertia terms and leads to linear governing equations, and the large Re leads to boundary layers whose thickness is much smaller than the cylinder radius, allowing one to neglect curvature terms in the sidewall boundary layer analysis. Of course, this raises the question as to what happens as the differential rotation is increased; how is the boundary layer structure altered given that the increased meridional flow driven by the Ekman layers may lead to a fundamental change in the boundary layer structure (e.g., see [4]), and at some finite strength of the differential rotation the nonlinear terms will become non-negligible and instabilities can be expected to ensue. Hart and Kittelman [5] provide some insights from flow visualization experiments in the case where the rapidly rotating cylinder has the top endwall rotating faster than the rest of the cylinder. Lopez [6] simulated this and other related flows solving the axisymmetric Navier–Stokes equations, and Lopez and Marques [7] investigated the three-dimensional instabilities of that flow. The boundary layer analysis [3] shows that the relative roles of the $Re^{-1/4}$ and $Re^{-1/3}$ layers are very different when the split in the

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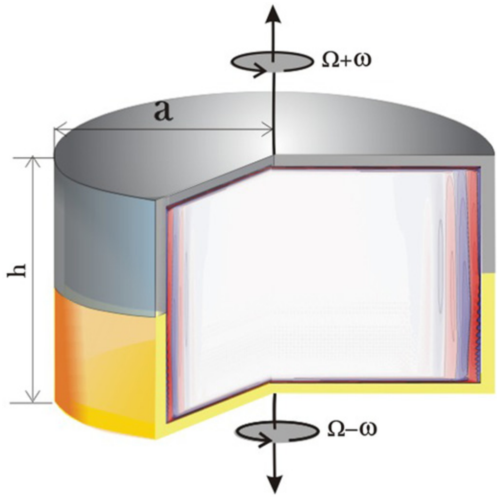

Fig. 1. Schematic of the flow system. The inset shows azimuthal vorticity contours of an axisymmetric time-periodic state at $Re = 10^5$, $Ro = 0.110$ and $\gamma = 1$.

cylinder is at mid-height compared to when it is at one of the corners where an endwall meets the sidewall. This difference raises the question as to how does the nonlinear behavior differ when the split is at half-height.

For these rapidly rotating split cylinder problems, in the absence of instabilities, the interior flow is in solid-body rotation with the mean rotation rate of the two cylinder halves. For fast enough mean rotation, disturbances from instabilities can only penetrate into the interior if their frequencies are less than twice the mean rotation frequency, and then only in the form of inertial wave beams along directions determined by their frequencies. In the inviscid limit, this is governed by the inertial wave dispersion relation [8], but for large but finite Re and finite differential rotation, viscous and nonlinear effects come into play, as well as mean-flow deformations leading to bulk flows that have non-constant angular speed. Furthermore, how these inertial wave beams feed back on the boundary layer and corner instabilities is not obvious, and we also try to address this.

2. Governing equations and numerical methods

Consider the flow in a circular cylinder of radius a and height h, completely filled with a fluid of kinematic viscosity ν. The cylinder is split in two, the top half rotates with angular speed $\Omega + \omega$ and the bottom half with angular speed $\Omega - \omega$. Fig. 1 shows a schematic of the flow.

The Navier–Stokes equations, non-dimensionalized using a as the length scale and $1/\Omega$ as the time scale, are

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + 1/Re\nabla^2\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u} = (u, v, w)$ is the velocity field in polar coordinates $(r, \theta, z) \in [0, 1] \times [0, 2\pi] \times [-\gamma/2, \gamma/2]$, and p is the kinematic pressure. There are three governing parameters:

$$\begin{aligned} &\text{Reynolds number } Re = \Omega a^2/\nu, \\ &\text{Rossby number } Ro = \omega/\Omega, \\ &\text{aspect ratio } \gamma = a/h. \end{aligned} \tag{2}$$

The boundary conditions are no-slip:

$$\begin{aligned} &z = 0.5\gamma: \quad (u, v, w) = (0, r(1 + Ro), 0), \\ &z = -0.5\gamma: \quad (u, v, w) = (0, r(1 - Ro), 0), \\ &r = 1, \quad z \in (0, 0.5\gamma): \quad (u, v, w) = (0, 1 + Ro, 0), \\ &r = 1, \quad z \in (0, -0.5\gamma): \quad (u, v, w) = (0, 1 - Ro, 0). \end{aligned} \tag{3}$$

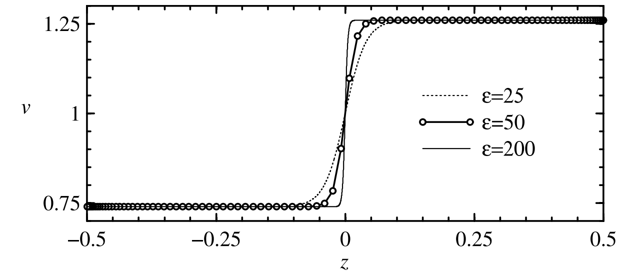

Fig. 2. Profiles of the regularized sidewall boundary condition for the azimuthal velocity, $v(z) = 1 + Ro\tanh(\epsilon z)$, with $Ro = 0.26$ and ϵ as indicated. The $\epsilon = 50$ case shows open symbols at the Chebyshev collocation points corresponding to $n_z = 100$.

In this present study, we only consider axisymmetric flows. The governing equations (1) have been solved using a second-order time-splitting method, with space discretized via Chebyshev collocation in r and z:

$$\boldsymbol{u}(r, z, t) = \sum_{n=0}^{2n_r+1}\sum_{m=0}^{n_z} \hat{\boldsymbol{u}}_{mn}(t)\Xi_n(r)\Xi_m(z), \tag{4}$$

where Ξ_n is the nth Chebyshev polynomial. The spectral solver is based on that described in Ref. [9] and it has been used extensively in a wide variety of enclosed cylinder flows. In the present paper where we focus on flows with $Re = 10^5$, we have typically used $n_r = 100$ and $n_z = 100$ and $\delta t = 4\times10^{-4}$. This provides sufficient spacial resolution to have between 6 and 9 collocation points in the bottom and top boundary layers (the two are not symmetric due to the differential rotation Ro), and δt is small enough to capture the dynamics.

The jump discontinuity in the sidewall boundary condition for the azimuthal velocity is problematic when solving the system using a spectral method as it leads to Gibb's phenomenon. This issue can be remedied by regularizing the boundary condition by smoothing out the jump over a small distance, in essentially the same way as the corner discontinuity between a sidewall and a differentially rotating endwall is regularized [10]. Specifically, we replace the boundary condition for the azimuthal velocity with

$$v(r = 1, z) = 1 + Ro\tanh(\epsilon z), \tag{5}$$

where ϵ governs the distance over which the jump is smoothed out. Fig. 2 shows the azimuthal velocity profile, η, at the sidewall for $Ro = 0.26$ and various values of ϵ. In the $\epsilon = 50$ case, the open symbols are at the Chebyshev collocation points corresponding to $n_z = 100$. Fig. 3 shows the azimuthal vorticity for $Re = 10^4$, $Ro = 0.26$, $\gamma = 1$ with $\epsilon = 50$ and $\epsilon = 200$. There is little difference in selecting $\epsilon > 50$, and for the rest of the results presented here, we fix $\epsilon = 50$.

3. Basic state

The interior flow of the basic state (BS) in a finite rotating cylinder split into two differentially rotating parts is essentially in solid-body rotation with approximately the mean rotation rate of the two cylinder halves. The top and bottom endwall boundary layers are Ekman like, and the only meridional flow in the interior is a weak axial flow from the slower (bottom) to the faster (top) rotating endwall. The interesting flow structure is associated with the sidewall boundary layer. Van Heijst [3] used boundary layer analysis to show that the main effect of the two endwalls is the order $Re^{-1/2}$ transport that appears in the sidewall Stewartson layer. He described how the combination of the $Re^{-1/4}$ and $Re^{-1/3}$ boundary layers matches the boundary conditions. For the case we study here, with the discontinuity in the middle of the cylinder,