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Electroosmotic flow of a power-law fluid in a slit microchannel with gradually varying channel height and wall potential

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ABSTRACT

A mathematical model based on the lubrication theory is presented for quasi-one-dimensional electroosmotic flow of a power-law fluid through a slit microchannel with undulating and non-uniformly charged walls. The channel height and the wall potential may vary periodically with axial position, with a wavelength much longer than the average channel height. Owing to the nonlinear rheology, the pressure gradient that is internally induced to satisfy continuity of flow has to be found numerically. A trial-and-error method is adopted to search for a flow rate that will give rise to an axial pressure gradient distribution with a zero average over one wavelength of the channel. When the flow behavior index is equal to the reciprocal of an integer, polynomial equations relating the flow rate and the local pressure gradient can be deduced, which will greatly facilitate the seeking of the solution by trial and error. Numerical results are also presented to illustrate how the flow behavior index may qualitatively change the combined effect of the geometric and electrokinetic wall patterns on the flow rate.

1. Introduction

In microfluidics, a fluid is often transported in a microchannel by means of an applied electric field. Such electrically driven flow, which draws upon the unbalanced charge distribution in the electric double layer (EDL) formed near a charged surface, is commonly called the electroosmotic (EO) flow. Since the pioneering work by Burgreen and Nakache [1] five decades ago, EO flow in microchannels has been extensively studied, especially in the past twenty years. These existing studies, whether theoretical or experimental, are mostly for EO flow of Newtonian fluids. Effects of non-Newtonian behaviors on EO flow have not received much attention until recently. The need for an in-depth understanding of non-Newtonian EO flow stems from the fact that microfluidic applications often involve complex fluids such as polymeric solutions and bio-fluids [2].

Das and Chakraborty [3] and Chakraborty [4] were among the first who presented theoretical models for EO flow of non-Newtonian fluids in microchannels. These authors adopted the power-law model to describe the non-Newtonian rheology. EO flows of other non-Newtonian fluids, such as Bingham [5], viscoelastic [6], and viscoplastic [7,8] materials, have also been investigated. To this date, the most chosen rheological model for non-Newtonian EO flow has been the power-law model; some typical works, among many others, are found in Refs. [9–15]. The power-law model, also known as the Ostwald–de Waele model, is a relatively simple two-parameter model, by which the shear-thinning, Newtonian, or shear-thickening behaviors can be conveniently represented by the flow behavior index being less than, equal to, or larger than unity, respectively.

Some noticeable analytical solutions for EO flow of power-law fluids in microchannels have been obtained by Yang and his collaborators [9,11,16,17]. One remarkable finding by these authors is an expression for the generalized Smoluchowski slip velocity for power-law fluids. For EO flow under the Debye–Hückel approximation (i.e., very small electric potentials), they also found closed-form analytical solutions for some particular values of the flow behavior index n (namely, $n = 1$, $1/2$ and $1/3$), and approximate analytical solutions for an arbitrary value of n. These and other analytical studies on EO flow of non-Newtonian fluids are, however, limited to uniform channels of simple geometry (e.g., a uniform parallel-plate or circular channel) such that the flow is steady and unidirectional, with the velocity depending on one transverse coordinate only. This is because, for such steady one-dimensional flow, the shear stress distribution can be determined *a priori*, which will then allow the velocity to be found straightforwardly by integration. For a flow behavior index n, the integrand has the form of a certain function raised to the exponent of $1/n$. If the function is an elementary function and n is the inverse of an integer, a

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closed-form analytical expression can be obtained for the velocity profile.

Non-uniformities are known to occur to microchannels. For example, the zeta potential, or the electric potential at the shear plane in the EDL, may vary spatially by construction or owing to unavoidable surface defects during fabrication. For Newtonian fluids, many have studied EO flow in microchannels with non-uniformly charged walls [18–22]. The problem is of practical importance because non-uniform wall charge may lead to secondary flow in the form of flow separation and recirculation. The problem becomes more interesting when non-uniform wall charge interacts with an undulating wall shape, as has been investigated by Ajdari [23,24]. He showed that the combined effect of periodic wall charge and shape modulation is to generate net flow even if the walls are on average electro-neutral. Charge modulation alone can only produce periodic convective cells, but will not generate net flow. Net effects may happen only when the symmetry of forward–backward flow induced by equal positive–negative charge distributions is broken by the superposition of a wavy wall. The net flow can be in a direction as if it were uniformly negatively charged even when the average wall charge is positive, and vice versa. Ghosal [25] also studied EO flow in channels where the cross-section and surface charge may vary slowly in the axial direction. These existing models are, however, for Newtonian fluids only. The desire for extending the work to non-Newtonian fluids has motivated the present study.

In this paper, we aim to study EO flow of a power-law fluid in a non-uniform slit microchannel with periodic axial variations of wall charge and channel height. The objective is to develop a model that enables us to examine the effects due to the power-law rheology on the interaction between the two wall patterns (one electrokinetic and one geometric) in controlling the flow through the channel. The flow is intrinsically two dimensional, and hence the analytical approach mentioned above will no longer be applicable. We shall nevertheless simplify the present model by means of the lubrication approximation, by which the problem can be formulated in a quasi-one-dimensional manner, thereby avoiding solving the momentum equations in full. The challenging part in the present problem is to determine an unknown pressure distribution along the channel. The pressure as a function of the axial coordinate, which is internally induced so as to maintain a constant flow rate through a channel with axial non-uniformities, has to be found numerically owing to the nonlinear interaction between the hydrodynamic and electric forcings for a non-Newtonian fluid. This distinguishes the present study from previous studies by the authors [26,27], which also look into EO flow of power-law fluid in a non-uniform channel, but are simplified by the use of the Newtonian Helmholtz–Smoluchowski slip boundary condition on taking into account a near-wall Newtonian depletion layer.

Our problem is defined in further detail in Section 2, where a mathematical formulation based on the lubrication approximation and the Debye–Hückel approximation is presented. Highly nonlinear equations are to be solved for the flow rate, which is a constant, and the pressure gradient distribution, which is a periodic function of the axial coordinate. To this end, a trial-and-error solution method is used, as described in Section 3. This method involves the searching for a flow rate that leads to a pressure distribution where the net pressure change over one wavelength is zero. For particular values of the flow behavior index n, namely equal to the reciprocal of an integer, polynomial equations can be derived to relate the flow rate and the local pressure gradient. When the integer is an even integer, the type of stress distribution has to be identified on deriving these polynomial equations. The availability of these analytical relationships will alleviate the numerical efforts involved in finding the solution. In Section 4, some physical discussion is presented. We shall look into how the flow behavior index

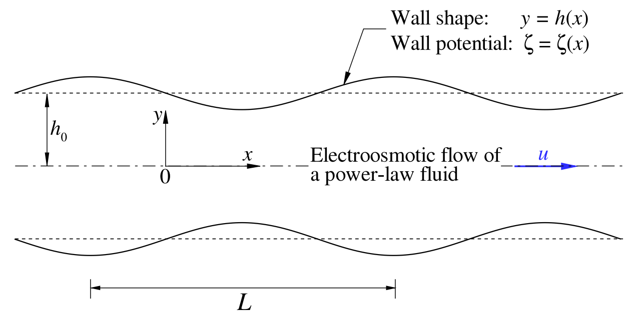

Fig. 1. Electroosmotic flow of a power-law fluid through a slit microchannel with undulating and non-uniformly charged walls, where (x, y) are the axial and transverse coordinates. The wall shape $y = h(x)$ and the wall potential $\zeta = \zeta(x)$ are periodic functions of x with a wavelength L, which is much longer than half the average channel height h_0.

may modify the combined effect of the wall undulation and the charge modulation on the EO flow. We shall show that the dependence of the flow rate on the wall pattern parameters may change with the flow behavior index, not only quantitatively, but also qualitatively.

2. Mathematical formulation

Our problem is to consider steady electroosmotic (EO) flow of a power-law fluid through a slit microchannel, of which the channel height as well as the wall potential may vary gradually and periodically in the streamwise direction. Fig. 1 shows a definition sketch of the problem, where (x, y) are the axial and transverse coordinates, and the x-axis is along the centerline of the channel. For simplicity, only flow that is symmetrical about the x-axis is considered: the upper/lower walls are at $y = \pm h(x)$ and the wall potential at either wall is given by $\zeta = \zeta(x)$, both being periodic functions of x with the same wavelength L. The wavelength L, which is the length scale for variations of velocity in the axial direction, is assumed to be much longer than the mean channel height: $L \gg h_0$. With this sharp contrast in length scales, we further assume that the Reynolds number of the flow is so small that the lubrication approximation [28,29] can be applied here. The flow is therefore nearly one dimensional: the axial velocity u is an order of magnitude larger than the transverse velocity v. Also, the inertia of the flow can be ignored, and the change of u in the x-direction is much milder than that in the y-direction.

In this work, the near-wall Newtonian depletion layer is assumed to be so thin that it is completely covered by the EDL, and therefore its effect on the bulk flow can be ignored. This will be valid when the thickness of the depletion layer, which is approximately the radius of gyration of the molecules making up the nonlinear rheology, is of the order of nanometers [30], while the thickness of the EDL, which depends on the bulk ion concentration, is of the order of hundreds of nanometers. On ignoring the depletion layer, the fluid is taken to be homogeneously non-Newtonian throughout the flow domain. The rheological behavior exhibited by a power-law fluid under simple shear is as follows:

$$\tau = \mu \left| \frac{du}{dy} \right|^{n-1} \frac{du}{dy} = \begin{cases} \mu \left(\dfrac{du}{dy} \right)^n & \text{for } \dfrac{du}{dy} > 0 \\ -\mu \left(-\dfrac{du}{dy} \right)^n & \text{for } \dfrac{du}{dy} < 0, \end{cases} \tag{1}$$

where τ is the shear stress, μ is the flow consistency, and n is the power-law or flow behavior index of the fluid. The shear-thinning, Newtonian, and shear-thickening behaviors are exhibited when