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Analytical solution of the transient electro-osmotic flow of a generalized fractional Maxwell fluid in a straight pipe with a circular cross-section

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ABSTRACT

In this study, we examine the transient electro-osmotic flow of a generalized Maxwell fluid with a fractional derivative in a narrow capillary tube. Using the integral transform method, analytical expressions are derived for the electric potential and transient velocity profile by solving the linearized Poisson–Boltzmann equation and the Navier–Stokes equation. We show that the distribution and establishment of the velocity comprises two parts: the steady and unsteady parts. We demonstrate the effects of the relaxation time, fractional derivative parameter, and the Debye–Hückel parameter on the generation of flow in a graphical manner and we analyze them numerically. The velocity overshoot and oscillation are observed and discussed.

1. Introduction

The research field of electro-osmosis has become increasingly attractive due to the development of microfluidic devices and their applications in microelectromechanical systems and microbiological sensors [1,2]. Recently, several studies [3,4] have indicated that the micelle structure of polymer electrolyte membranes might only comprise cylindrical nano-channels that facilitate water and proton transport, rather than large water pore clusters connected by smaller nano-channels, as found in Gierke's model. This raises the problem of how to model the electro-osmotic flow of fluids in a straight pipe with a circular cross-section.

Most previous theoretical studies of electro-osmotic flow were limited to a fully developed steady-state flow [5–8]. An electro-osmotic flow problem in an infinite cylindrical pore with a uniform surface charge density was studied analytically by Berg and Ladipo [9], where the results demonstrated the distribution of the electric potential and the counter-ions (protons), the velocity profile of the water flow and its associated total flux, as well as the protonic current, conductivity, and water drag. Chang [10] presented a theoretical study of the transient electro-osmotic flow through a cylindrical microcapillary containing a salt-free medium with both a constant surface charge density and a constant surface potential, where the exact solutions for the electric potential distribution and transient electro-osmotic flow velocity were derived by solving the nonlinear Poisson–Boltzmann equation and the Navier–Stokes equation. By applying a stepwise voltage, Mishchuk and González-Caballero studied a theoretical model of electro-osmotic flow in a wide capillary [11], where both periodic and aperiodic flow regimes were studied with arbitrary pulse/pulse or pulse/pause durations and amplitudes.

In general, microfluidic devices are used to analyze biofluids, which are often solutions of long chain molecules and their behavior is very different from that of a Newtonian fluid, including memory effects, normal stress effects, and yield stress. Thus, these fluids cannot be treated as Newtonian fluids. Recently, many researchers have focused on the non-Newtonian fluid behavior of biofluids in electrokinetically driven microflows. The first study of non-Newtonian effects in an electro-osmotic flow was reported by Das and Chakraborty [12,13], where they treated biofluids as power-law fluids and they obtained an analytical solution to describe the transport characteristics of a non-Newtonian fluid flow in a rectangular microchannel under the sole influence of electrokinetic effects. For the same non-Newtonian fluid model, Zhao and Yang [14] obtained the general Smoluchowski velocity for electro-osmosis over a surface with arbitrary

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zeta potentials. Park and Lee [15] derived a semi-analytical expression for the Helmholtz–Smoluchowski velocity under pure electro-osmosis conditions for the full Phan–Thien–Tanner (PTT) constitutive equation and they used a finite volume method to numerically calculate the flow of the full PTT model in a rectangular duct under the action of electro-osmosis and a pressure gradient [16].

In the present study, we model the non-Newtonian behavior of biofluids using the generalized Maxwell fluid with a fractional derivative. The aim of this study is to present the analytical solution for the unsteady electro-osmotic flow of a generalized Maxwell fluid in a cylindrical capillary, and we also discuss the effects of physical parameters on the generation of flow, such as the relaxation time, fractional derivative parameter, and the Debye–Hückel parameter.

2. Governing equations

2.1. Constitutive equation for a generalized fractional Maxwell fluid

The Maxwell model describes one of the simplest linear viscoelastic fluids, where it comprises a branch made of a spring in series with a dashpot. The spring is the elastic element and because the force is proportional to the extension, it represents a perfectly elastic body that obeys Hooke's law. The dashpot is the viscous element and the force is proportional to the rate of extension, so it represents a perfectly viscous body that obeys Newton's law. The corresponding constitutive equation can be expressed as

$$\tau(t)+\lambda_r\frac{d\tau}{dt}\tau=\mu\frac{d\gamma}{dt}, \tag{1}$$

where τ is the shear stress, γ is the shear strain, $\lambda_r=\mu/G_0$ is the relaxation time for which G_0 is a shear modulus, and μ is a viscosity constant.

In recent decades, fractional calculus has been utilized with much success in the description of complex dynamics such as relaxation, wave, and viscoelastic behaviors. Due to the development of the operator in fractional calculus, a straightforward method for introducing fractional derivatives into models of linear viscoelasticity is to replace the first derivative in the constitutive equation of the Maxwell model with a fractional derivative of order $\alpha\in(0,1)$. At a physical level, Bagley and Torvik [17] demonstrated that the theory of viscoelasticity for coiling polymers predicts constitutive relations with fractional derivatives. Subsequently, Makris et al. [18] proposed a generalized Maxwell model with a fractional derivative, where a very good fit with the experimental data was achieved when the first-order derivatives of the Maxwell model were replaced by fractional-order derivatives [19]. The shear stress–strain relationship in the fractional derivative Maxwell model proposed by Makris et al. [18] is

$$\tau+\lambda_r^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}\tau=G_0\lambda_r^{\beta}\frac{d^{\beta}}{dt^{\beta}}\gamma, \tag{2}$$

where α and β are fractional parameters such as $0\le\alpha,\beta\le1$, and d^{α}/dt^{α} is the Caputo fractional derivative defined as

$$\frac{d^{\alpha}}{dt^{\alpha}}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau,\quad 0\le\alpha<1. \tag{3}$$

However, Friedrich [20] proved that this type of rheological constitutive equation exhibits fluid-like behavior only in the case where $\beta=1$. Therefore, the following constitutive equation for the generalized Maxwell fluid is used in the present study:

$$\tau+\lambda_r^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}\tau=\mu\frac{d\gamma}{dt}. \tag{4}$$

2.2. Mathematical model of the flow

We consider the electro-osmotic flow of a generalized Maxwell fluid with a dielectric constant ε at rest at time $t\le0$, which is contained in a straight pipe with a circular cross-section and radius R. It is assumed that the pipe wall is uniformly charged with a zeta potential, ψ_w. When an external electric field E_0 is imposed along the axial direction, the fluid in the pipe is set in motion due to electro-osmosis.

All quantities are referred to cylindrical polar coordinates (r,θ,z), where r is measured from the axis of the pipe and z along its length. If we assume a velocity distribution of the form

$$(0,0,u(r,t)),\quad 0\le r\le R,\ t>0, \tag{5}$$

the initial condition is given by

$$u(r,0)=0,\quad 0\le r\le R \tag{6}$$

and the equation of continuity $\nabla\cdot\mathbf{V}=0$ is satisfied automatically.

According to the theory of electrostatics, the net charge density ρ_e is expressed by a potential distribution ψ, which is given by the Poisson equation,

$$\nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}=-\frac{\rho_e}{\varepsilon}. \tag{7}$$

The boundary condition is that the zeta potential ψ_w is given on the wall of the pipe,

$$\psi(R,\theta)=\psi_w,\qquad \left.\frac{\partial\psi}{\partial r}\right|_{r=0}=0. \tag{8}$$

In the present study, we assume that the charge distribution in the Debye layer is not affected by time, i.e., the wall of the pipe has a constant electric potential E_0. Thus, the relevant equation of motion reduces to

$$\rho\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})-\rho_eE_0 \tag{9}$$

which has the following initial and boundary conditions

$$u(r,0)=\left.\frac{\partial u}{\partial t}\right|_{t=0}=0, \tag{10}$$

$$u(r,t)=0,\quad r=R. \tag{11}$$

3. Exact solution of the model

For small electrical potential values ψ of the electrical double layer (EDL), the Debye–Hückel approximation can be used, which means that the electrical potential is physically small compared with the thermal energy of the charged species. Thus, we have the linearized charge density

$$\rho_e=-\frac{2z_v^2e^2n_0\psi}{k_BT} \tag{12}$$

where z_v is the valence of ions, e is the fundamental charge, k_B is the Boltzmann constant, and T is the absolute temperature.

Using the Debye–Hückel approximation [21,22], Eq. (7) can be linearized to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)=\kappa^2\psi. \tag{13}$$

The equation of motion (9) then becomes

$$\rho\frac{\partial u}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})-\kappa^2\varepsilon\psi E_0, \tag{14}$$