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Stokes flow through a twisted tube with square cross-section

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ABSTRACT

Flow through a twisted tube with square cross-section and helical corrugations of arbitrary pitch is computed under conditions of Stokes flow. The governing equations are formulated in non-orthogonal helical coordinates in terms of a coupled system of linear differential equations describing the longitudinal and transverse flow over the tube cross-section. Numerical solutions are computed by a finite-difference method on a staggered grid incorporating pressure nodes and three sets of velocity nodes. The results illustrate the structure of the secondary flow developing over the tube cross-section and document the effect of the helical corrugations on the axial flow rate.

1. Introduction

Partitions, fins, ridges and ribs are inserted in channels and tubes to disturb the otherwise rectilinear streamlines of pressure-driven channel or pipe flow and thereby initiate a rotational motion that promotes passive mixing and facilitates convective scalar transport. Applications can be found in high- and low-speed heat exchangers and in processing equipment encountered in the chemical and food industries. A generalized engineering concept employs tubes with helical corrugations generated either by twisting a straight tube with non-circular cross-section, such as a tube with elliptical, sinusoidal, or rectangular cross-section, or by embossing helical corrugations on a circular tube. These designs are especially attractive in microfluidics applications and other miniaturized devices where the details of the boundary geometry are of paramount importance and sometimes the only available control parameter for manipulating the flow (e.g., [1]).

Twisted tubes should be distinguished from spirally coiled tubes where a secondary flow induced by curvature, known as the Dean flow, arises. Flow through a chain of pipe bends has been shown to exhibit chaotic advection in the context of Dean's asymptotic solution [2]. Although flow through commercial and laboratory tubes with helical corrugations have been investigated in the laboratory with reference to enhanced heat transfer in high-Reynolds-number flow, a satisfactory hydrodynamic analysis is not available. Since unidirectional flow is established in the two diametrically opposite limits of zero or infinite pitch, an optimal pitch is expected where the bulk rotation of the fluid in the core becomes strongest.

Wang [3] carried out a perturbation analysis for tubes with small-amplitude sinusoidal corrugations of arbitrary pitch in low-Reynolds-number inertial flow and confirmed the existence of an optimal pitch. Pozrikidis [4] performed a complementary perturbation expansion for Stokes flow in tubes with arbitrary cross-section in the limit of large pitch, and presented numerical solutions obtained by a finite-element method for an assortment of geometries up to second-order in the helical wave number. In more recent years, numerical solutions of the equations governing hydrodynamics and convective heat transport in twisted tubes with elliptical or rectangular cross-section were presented in the applied engineering literature based on commercial codes (e.g., [5,6]). In these typical CFD calculations, the simplifications stemming from the helical are not exploited. Instead, full domain discretization of a test section with chosen inlet and outlet conditions is employed.

In this paper, numerical solutions of the equations governing Stokes flow through a twisted tube with square cross-section at arbitrary helical wave numbers are presented. The governing equations are formulated in non-orthogonal helical coordinates and then solved over the square tube cross-section using a novel finite-difference method on a staggered grid. The numerical results extend and corroborate previous asymptotic results for large pitch, and confirm that the helical pitch is an important parameter of the motion. Venues for complementary experimental work are outlined in the Discussion.

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2. Problem statement

We consider Stokes flow through a corrugated tube that arises by twisting a straight tube with arbitrary cross-section around its axis over the length of the pitch, L. A point on the surface of the tube is identified by its cylindrical polar coordinates, (x, σ, φ), where the axial position, x, and azimuthal angle, φ, are regarded as independent variables. The distance of a point on the tube surface from the tube center is given by

$$\sigma = \Sigma(n\varphi - kx), \tag{1}$$

where $\Sigma(w)$ is a shape function with period 2π, the integer n is the azimuthal wave number for a tube with n-fold rotational cross-sectional symmetry, and the real number k is the axial wave number corresponding to the axial wave length or pitch, $L = 2\pi/k$. Physically, the helical geometry arises by twisting the tube cross-section at a particular location by an angle that depends linearly on the axial distance, x. Note that the tube cross-sectional geometries at positions x and $x + L$ are identical.

A twisted tube with a square cross-section corresponding to $n = 4$ is shown in Fig. 1. In practice, the tube can be fabricated by a three-dimensional printer or else by stacking thin plates perforated by square holes while rotating the plates by an angle determined by the pitch.

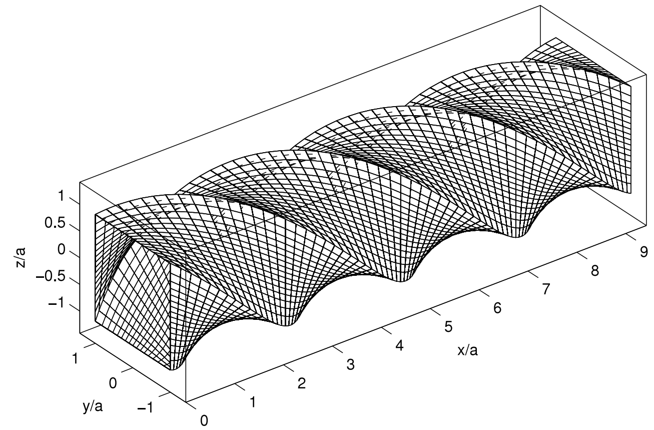

Fig. 1. Depiction of a helically twisted tube with a square cross-section of side length $2a$ for helical pitch $L = 3\pi a/4$, corresponding to helical wave number $k = 2/3$ and $\alpha = 1/6$.

2.1. Helical coordinates

A point inside the tube can be identified by its nonorthogonal helical curvilinear coordinates, $(\hat\sigma, \hat\varphi, \hat x)$, as shown in Fig. 2. The helical coordinates are related to the underlying Cartesian coordinates by

$$x = \hat x, \qquad y = \hat\sigma \, \cos(\hat\varphi + \alpha \, \hat x), \qquad z = \hat\sigma \, \sin(\hat\varphi + \alpha \, \hat x), \tag{2}$$

where $\alpha \equiv k/n = 2\pi/(nL)$ is a wave number. The range of variation of the helical coordinates inside the tube over the length of one pitch is

$$0 < \hat\sigma < \Sigma(n\hat\varphi), \qquad 0 < \hat\varphi < 2\pi, \qquad 0 < \hat x < L. \tag{3}$$

Lines of constant $\hat\varphi$ on the tube surface correspond to a constant tube radius, $\hat\sigma$, as illustrated in Fig. 1.

The covariant metric tensor of the helical coordinates is

$$g_{ij} = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat\sigma^2 & \alpha\,\hat\sigma^2 \\ 0 & \alpha\,\hat\sigma^2 & 1+\alpha^2\,\hat\sigma^2 \end{bmatrix}, \tag{4}$$

where $\xi^1 = \hat\sigma$, $\xi^2 = \hat\varphi$ and $\xi^3 = \hat x$.

A mean pressure gradient along the tube axis drives a three-dimensional pressure-driven flow. The fluid velocity can be resolved into components corresponding to the helical coordinates $(\hat\sigma, \hat\varphi, \hat x)$,

$$\mathbf{u} = u_{\hat\sigma}\,\mathbf{e}_{\hat\sigma} + u_{\hat\varphi}\,\mathbf{e}_{\hat\varphi} + u_{\hat x}\,\mathbf{e}_{\hat x}, \tag{5}$$

where

$$\mathbf{e}_{\hat\sigma} = \frac{1}{\sqrt{g_{11}}} \left(\frac{\partial \mathbf{x}}{\partial \hat\sigma}\right)_{\hat x,\hat\varphi}, \qquad \mathbf{e}_{\hat\varphi} = \frac{1}{\sqrt{g_{22}}} \left(\frac{\partial \mathbf{x}}{\partial \hat\varphi}\right)_{\hat x,\hat\sigma}, \qquad \mathbf{e}_{\hat x} = \frac{1}{\sqrt{g_{33}}} \left(\frac{\partial \mathbf{x}}{\partial \hat x}\right)_{\hat\sigma,\hat\varphi} \tag{6}$$

are position-dependent nonorthogonal unit vectors.

The cylindrical polar velocity components are related to the corresponding helical components by

$$u_\sigma = u_{\hat\sigma}, \qquad u_\varphi = u_{\hat\varphi} + \frac{\alpha\,\hat\sigma}{\sqrt{1+\alpha^2\hat\sigma^2}}\, u_{\hat x} = u_{\hat\varphi} + \alpha\,\hat\sigma\, u_x, \qquad u_x = \frac{1}{\sqrt{1+\alpha^2\hat\sigma^2}}\, u_{\hat x}. \tag{7}$$

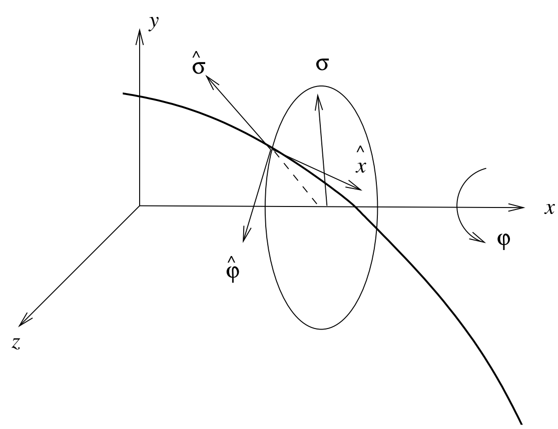

Fig. 2. Definition of helical coordinates, $(\hat x, \hat\sigma, \hat\varphi)$, in relation to the global Cartesian coordinates and polar cylindrical coordinates, (x, σ, φ).

A key observation is that, if the flow is helically symmetric along the entire length of the tube, these velocity components are independent of $\hat x$, and only depend on $\hat\sigma$ and $\hat\varphi$. Moreover, since the flow is assumed to be fully developed, the axial derivative of the pressure is a constant,

$$-\left(\frac{\partial p}{\partial \hat x}\right)_{\hat\sigma,\hat\varphi} \equiv G, \tag{8}$$

where G is the negative of the streamwise pressure gradient.

3. Governing equations

The governing equations in the helical coordinates presented in Section 2 were derived by Tung & Laurence [7] and discussed by Pozrikidis [4]. In this section, the equations are recast in a form that is suitable for the implementation of numerical methods.

We will work in Cartesian coordinates in a plane of constant streamwise position, $\hat x$, defined such that

$$\hat y = \hat\sigma \, \cos\hat\varphi, \qquad \hat z = \hat\sigma \, \sin\hat\varphi. \tag{9}$$

3.1. Continuity equation

The continuity equation in a plane normal to the x axis corresponding to a fixed value of $\hat x$ takes the familiar form

$$\hat\nabla \cdot \hat{\mathbf{u}} = \frac{1}{\hat\sigma} \left(\frac{\partial(\hat\sigma\, u_{\hat\sigma})}{\partial\hat\sigma} + \frac{\partial u_{\hat\varphi}}{\partial\hat\varphi} \right) = 0, \tag{10}$$