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The structural sensitivity of open shear flows calculated with a local stability analysis

Matthew P. Juniper [a,*], Benoit Pier [b]

[a] Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK

[b] Laboratoire de mécanique des fluides et d'acoustique, École centrale de Lyon - CNRS - Université Claude-Bernard Lyon 1 - INSA Lyon, 36 avenue Guy-de-Collongue, 69134, Écully, France

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ABSTRACT

The structural sensitivity shows where an instability of a fluid flow is most sensitive to changes in internal feedback mechanisms. It is formed from the overlap of the flow's direct and adjoint global modes. These global modes are usually calculated with 2D or 3D global stability analyses, which can be very computationally expensive. For weakly non-parallel flows the direct global mode can also be calculated with a local stability analysis, which is orders of magnitude cheaper. In this theoretical paper we show that, if the direct global mode has been calculated with a local analysis, then the adjoint global mode follows at little extra cost. We also show that the maximum of the structural sensitivity is the location at which the local k^+ and k^- branches have the same imaginary value. Finally, we use the local analysis to derive the structural sensitivity of two flows: a confined co-flow wake at $Re = 400$, for which it works very well, and the flow behind a cylinder at $Re = 50$, for which it works reasonably well. As expected, we find that the local analysis becomes less accurate when the flow becomes less parallel.

1. Introduction

Many open flows have a steady solution to the Navier–Stokes equations that becomes unstable above a critical Reynolds number. Usually this instability is driven by one region of the flow, which is called the wavemaker region. The rest of the flow merely responds to forcing from this region. The shape, linear growth rate, and frequency of the instability can be calculated by considering the evolution of small perturbations about the steady solution. This is known as the direct global mode. The direct global mode emanates from the wavemaker region and grows spatially downstream, reaching a maximum at the streamwise location where the spatial growth rate is zero. For example, in the case of the flow behind a cylinder, this direct global mode is a sinuous flapping motion, whose nonlinear development is the familiar Kármán vortex street [1].

The receptivity of the direct global mode to harmonic open loop forcing is given by the last term in Eq. (9) of Ref. [2] and Eq. (7) of Ref. [3]. This term is proportional to the adjoint global mode, which is calculated in the same way as the direct global mode, but from the adjoint (rather than direct) linearized Navier–Stokes equations. If the perturbation magnitude is measured by the perturbation kinetic energy, which is the conventional approach, then there are only two significant differences between the direct and adjoint equations [2,4]. The first is the sign of the convection term, $V_j \partial v_i / \partial x_j$, and is called *convective non-normality*. The second is the appearance of a transconjugate operator, $v_j \partial V_j / \partial x_i$, and is called *component-type* non-normality, For the flows in this paper, the non-normality is almost entirely convective [4]. In a manner analogous to the direct global mode, the adjoint global mode emanates from the wavemaker region but grows spatially upstream, reaching a maximum at the streamwise location where the adjoint spatial growth rate is zero, or when it meets the upstream boundary. Physically, this reflects the fact that an open loop forcing signal will have most influence on the flow if it impinges on the wavemaker region, and if it is amplified by the flow before it does so.

The sensitivity of the direct global mode to changes in the linearized Navier–Stokes (LNS) equations is given by the penultimate term in Eq. (9) of Ref. [2]. This term is proportional to the overlap between the direct and adjoint global modes and is known as the structural sensitivity. It is equivalent to the sensitivity of the direct global mode to closed-loop feedback between the perturbation and the governing equations in the special case where the sensor and actuator are co-located. For example, in the case of the

* Corresponding author. Tel.: +44 1223 332 585.
E-mail address: mpj1001@cam.ac.uk (M.P. Juniper).

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flow behind a cylinder, it can quantify the sensitivity of the flow to the presence of a small control cylinder that produces a small force on the flow in the opposite direction to the velocity perturbation [5,6]. Given that the direct global mode grows downstream of the wavemaker region and that the adjoint global mode grows upstream, the structural sensitivity is clearly maximal in the wavemaker region itself. Indeed, the wavemaker region is often defined as the position of maximum structural sensitivity, although alternative definitions exist [4, Section 4.2.1]. Physically, this reflects the fact that, for a closed loop feedback mechanism to be effective, it requires firstly that the perturbation has significant amplitude at that point, which is quantified by the direct global mode, and secondly that the flow has significant receptivity at that point, which is quantified by the adjoint global mode.

The above concepts were first introduced for the flow behind a cylinder at $Re = 50$ by Hill [5] and Giannetti and Luchini [6,7] and have been extended to include the sensitivity to steady forcing and modifications to the base flow [8–10]. They have also been applied to recirculation bubbles [11] bluff bodies, both incompressible [12] and compressible [13], backward-facing steps [14], forward-facing steps [15], confined wakes [16,17], and a recirculation bubble in a swirling flow [18].

The direct global mode is usually found with a global stability analysis. This typically proceeds in three steps: (i) the Navier–Stokes (N–S) equations are linearized around a steady laminar flow, which is called the base flow and which is usually unstable; (ii) the equations are discretized and expressed as a 2D or 3D matrix eigenvalue problem; (iii) the most unstable eigenmodes are calculated with an iterative technique, such as an Arnoldi algorithm or power iteration. Each eigenmode consists of a complex eigenvalue, which describes the frequency and growth rate, and an eigenfunction, which describes the 2D or 3D shape that grows on top of the base flow until nonlinear effects become significant. As more elaborate configurations are examined, the number of degrees of freedom rapidly approaches millions, so global stability analyses can be extremely computationally expensive [4].

If the base flow varies slowly in the streamwise direction then the global stability analysis can be replaced with a local stability analysis [19]. The WKBJ approximation reduces the LNS equations over the entire domain into a series of local LNS or Orr–Sommerfeld (O–S) equations at each streamwise location. Each local equation can be discretized and expressed as a small matrix eigenvalue problem, which represents the dispersion relation between the complex frequency, ω, and the complex wavenumber, k. At each streamwise location, the value of ω is found for which the dispersion relation is satisfied and for which $\mathrm{d}\omega/\mathrm{d}k = 0$. This is known as the absolute complex frequency, ω_0 and its imaginary part, ω_{0i}, is the absolute growth rate. The flow is absolutely unstable in regions in which ω_{0i} is positive. These regions exist in every flow that is globally unstable due to hydrodynamic feedback. The frequency and growth rate of the linear global mode can be derived from the streamwise distribution of ω_0. This also gives a specific spatial position for the region of the flow that, in the context of the local analysis, is known as the wavemaker [20]. Local stability analyses are much quicker and require much less computer memory than global stability analyses because they convert one large matrix eigenvalue problem into several small independent matrix eigenvalue problems. This is why they have been used so widely in the past and why they are still used for flows that are beyond the range of global analyses [21–23].

In all existing papers, the adjoint global mode is calculated with a global stability analysis. The purpose of this paper is to show that, if a local stability analysis is used to calculate the direct global mode, then the adjoint global mode follows at almost no extra cost. This means that, for weakly nonparallel flows, adjoint global modes and structural sensitivities can be estimated quickly and cheaply, without deriving the adjoint equations. After defining the form of the direct and adjoint equations in Section 2, we derive this result rigorously in Section 3 for the Ginzburg–Landau equation (G–L), which is often used as a simple model for slowly-developing flows. We then apply this to the linearized N–S equations in Section 4 and demonstrate this on two flows in Section 5: a slowly-developing confined wake, and the flow behind a cylinder at $Re = 50$.

2. General form of the direct and adjoint equations

Many different conventions are used to describe direct and adjoint global modes. The convention used here is similar to that used for local stability analysis, so that it is easy to compare the local and global approaches. It differs from that used in Hill [5,24] and Giannetti and Luchini [6] in three ways. The direct and adjoint governing equations (1) and (2) have the same form so that their k^+ and k^- branches in the local analysis have the same physical meaning. The adjoint variables are denoted with †, rather than + or ∗, so that they are not confused with the k^+ branch or with the complex conjugate. The inner product contains a complex conjugate so that the inner product of a complex state variable with itself is a real number.

The linearized governing equations are expressed in terms of the direct state variable, $\psi(x, t)$, the adjoint state variable, $\psi^\dagger(x, t)$, the direct linear spatial operator L, and the adjoint linear spatial operator $\mathrm{L}^\dagger$:

$$\frac{\partial \psi}{\partial t} - \mathrm{L}\psi = 0, \tag{1}$$

$$\frac{\partial \psi^\dagger}{\partial t} - \mathrm{L}^\dagger\psi^\dagger = 0. \tag{2}$$

(The relationship between the direct and adjoint quantities will be specified in (8), after the inner product (7) has been defined.) Solutions to the initial value problems defined by (1) and (2) can be expressed for $t \in [0, \infty)$ as the sum of the direct and adjoint global modes:

$$\psi(x, t) = \sum_m \hat{\psi}_m(x) \exp(-\mathrm{i}\omega_m t), \tag{3}$$

$$\psi^\dagger(x, t) = \sum_n \hat{\psi}_n^\dagger(x) \exp(-\mathrm{i}\omega_n t). \tag{4}$$

Substituting (3) into (1) and (4) into (2) gives, for each mode,

$$-\mathrm{i}\omega_m\hat{\psi}_m - \mathrm{L}\hat{\psi}_m = 0, \tag{5}$$

$$-\mathrm{i}\omega_n\hat{\psi}_n^\dagger - \mathrm{L}^\dagger\hat{\psi}_n^\dagger = 0. \tag{6}$$

An inner product between state variables f and g is defined as

$$\langle f, g\rangle \equiv \int_{-\infty}^{+\infty} f^* g \,\mathrm{d}x. \tag{7}$$

If boundary terms are assumed to be zero, as in Giannetti and Luchini [6], Hill [5], then the relationship between the direct operator, L, and its adjoint, $\mathrm{L}^\dagger$, is given by

$$\langle \mathrm{L}\hat{\psi}_m, \hat{\psi}_n^\dagger\rangle = \langle \hat{\psi}_m, \mathrm{L}^\dagger\hat{\psi}_n^\dagger\rangle. \tag{8}$$

These definitions determine the relationship between ω_m and ω_n:

$$\langle \mathrm{L}\hat{\psi}_m, \hat{\psi}_n^\dagger\rangle = \langle \hat{\psi}_m, \mathrm{L}^\dagger\hat{\psi}_n^\dagger\rangle, \tag{9}$$

$$\langle -\mathrm{i}\omega_m\hat{\psi}_m, \hat{\psi}_n^\dagger\rangle = \langle \hat{\psi}_m, -\mathrm{i}\omega_n\hat{\psi}_n^\dagger\rangle, \tag{10}$$

$$\mathrm{i}\omega_m^*\langle \hat{\psi}_m, \hat{\psi}_n^\dagger\rangle = -\mathrm{i}\omega_n\langle \hat{\psi}_m, \hat{\psi}_n^\dagger\rangle, \tag{11}$$

$$(\omega_m^* + \omega_n)\langle \hat{\psi}_m, \hat{\psi}_n^\dagger\rangle = 0. \tag{12}$$

This is the bi-orthogonality condition: every adjoint mode is orthogonal to every direct mode, except for the pairs that satisfy $\omega_n = -\omega_m^*$.