European Journal of Mechanics B/Fluids 46 (2014) 97–108

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

# Stability of periodic gravity waves in the presence of surface tension

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ARTICLE INFO

*Article history:*
Received 18 September 2013
Received in revised form
3 January 2014
Accepted 14 February 2014
Available online 22 February 2014

*Keywords:*
Stability
Water waves
Surface tension
Gravity

ABSTRACT

The goal of this work is to investigate the effect of the inclusion of small surface tension on the instabilities of periodic gravity water waves that are present even in shallow water (Deconinck and Oliveras, 2011). Using the recent reformulation of Ablowitz, Fokas and Musslimani (2006), we compute periodic traveling water waves where the effects of both gravity and small surface tension are incorporated. The spectral stability of these solutions is examined using Hill's method (Deconinck and Kutz, 2006). It is found that the instabilities are not suppressed by the inclusion of surface tension. In fact, the growth rates associated with them increase as the surface tension grows. Generalizing the work of MacKay and Saffman (1986), the persistence of these instabilities is confirmed analytically for waves of small amplitude.


## 1. Introduction

The classical water wave problem is the problem of determining the shape and dynamics of the free surface on an incompressible, inviscid fluid. If, in addition, the fluid is irrotational, a velocity potential may be introduced. For one dimensional surface waves, the problem is described by the classical equations [1]

$$\begin{cases} \phi_{xx}+\phi_{zz}=0, & (x,z)\in D,\\ \phi_z=0, & z=-h,\ x\in(0,L),\\ \eta_t+\eta_x\phi_x=\phi_z, & z=\eta(x,t),\ x\in(0,L),\\ \phi_t+\dfrac{1}{2}\left(\phi_x^2+\phi_y^2\right)+g\eta & \\ \quad =\sigma\dfrac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}, & z=\eta(x,t),\ x\in(0,L), \end{cases} \tag{1}$$

where $h$ is the height of the fluid, $g$ is the acceleration due to gravity and $\sigma > 0$ is the coefficient of surface tension.[1] Further, $\eta(x,t)$ is the elevation of the fluid surface, and $\phi(x,z,t)$ is its velocity potential. In this paper, we focus on solutions on a periodic domain $D=\{(x,z)\mid 0\le x<L, -h<z<\eta(x,t)\}$; see Fig. 1.

The work presented here follows that of Deconinck and Oliveras [3]. They presented a thorough numerical overview of the spectral instabilities of periodic traveling one-dimensional gravity (i.e., $\sigma=0$) water waves. An emphasis of that work is the presence of oscillatory instabilities even for waves in shallow water ($kh < 1.363$, see [4,5], here $k=2\pi/L$). Since the underlying waves are periodic, their stability analysis uses Hill's method, see [6], which incorporates the conclusions from Floquet's Theorem with Fourier analysis. This associates with each wave a range of Floquet exponents $\mu$ which may be taken as $(-\pi/L,\pi/L]$. The growth rates of the oscillatory instabilities are small, even for waves of moderate amplitude, and the range of Floquet exponents with which they are associated is narrow (on the order of $10^{-4}$ for $L=2\pi$). A naive uniform distribution on $(-\pi/L,\pi/L]$ of Floquet exponents is bound to miss the presence of these instabilities, unless an exorbitantly large number of $\mu$ values are considered. Numerically, this is prohibitively expensive (often, no more than 100 $\mu$-values are chosen), and an adaptive approach is used in [3], with more values of $\mu$ considered near those values of the Floquet exponents where instabilities may arise, as predicted by MacKay and Saffman [7].

Our goal is to investigate the effect of the inclusion of surface tension on the oscillatory instabilities. It is well known that the incorporation of capillary effects leads to the presence of resonances in the Fourier representation of the periodic traveling water waves. If the resonance condition $R(\sigma,g,h,L)=0$ is satisfied, so-called Wilton ripples are found [1,8]. Even when $R(\sigma,g,h,L)\neq 0$, its value can be made arbitrarily small by the consideration of Fourier modes with sufficiently high wave number. This results in the presence of small denominators in the Stokes expansion of the wave profile. This is especially problematic for waves of moderate or high amplitude, whose accurate Fourier representation requires more modes. This is discussed in more detail in Sections 4 and 5. Because of this, we limit our investigations to the instabilities of waves of small amplitude, so that (near-) resonance is avoided. Waves in both shallow and deep water are considered.

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[1] As noted in [2], $\sigma > 0$ for liquid–gas interfaces.

http://dx.doi.org/10.1016/j.euromechflu.2014.02.010
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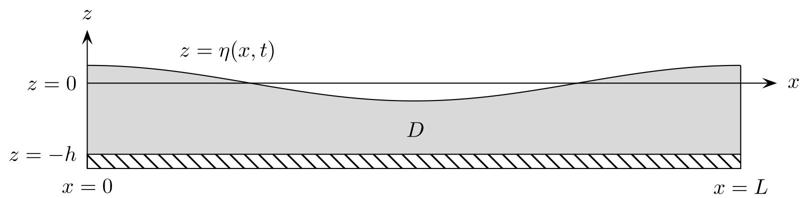


**Fig. 1.** The domain on which we solve Euler's equations.

In the next section we provide an overview of the literature on this classical problem. Section 3 discusses the reformulation of the water wave problem we use, both for the computation of the traveling wave solutions and for the analysis of their stability. After that, different sections are devoted to the computation of the solutions and to the numerical investigation of their spectral stability. In addition, we revisit the work of MacKay and Saffman [7], which allows for an analytical prediction of which modes may lead to instabilities. We finish with conclusions.

## 2. Literature overview

The study of water waves goes back as far as Newton (1687), Euler (1761) and Bernoulli (1738) [9]. The study of water waves benefits from theoretical contributions in addition to experimental ones. This literature review attempts to cover the literature that is most relevant to the current work. It is by no means comprehensive. First, we discuss the history of the computation of traveling wave solutions to (1). Next, we review the literature on the investigation of their stability properties.

Stokes was the first to construct solutions to Euler's equations in 1847. He introduced a form for a graph of a traveling wave on a periodic domain [10]. This was done perturbatively by adding successive harmonics of a cosine profile. In 1880, he conjectured there is a gravity wave of maximum height that is achieved when the distance from crest to trough is 0.142 wavelengths [9]. The first papers to show that series expansion in powers of the wave amplitude (or Stokes expansions) converges were due to Nekrasov (1921) [11] and Levi-Civita (1925) [12]. They showed that the Stokes series converges when the ratio of amplitude to wavelength is sufficiently small and the waves are in infinitely deep water. Struik (1926) [13] extended this analysis for water of finite depth.

Examining periodic surface gravity–capillary waves using an expansion like the one used by Stokes, Wilton (1915) [8] computed successive coefficients, while including the effects of surface tension. He showed that if the coefficient of surface tension in deep water is proportional to the inverse of an integer, the denominator of the expansion coefficients becomes zero. Since the terms of the series are computed only up to a scaling, he postulated that by choosing this scaling constant proportional to the vanishing denominator, the convergence of the series may once again be achieved.

Following Stokes's conjecture of a wave of greatest height for gravity waves [10], Crapper [14] investigated the possibility of a wave of maximum height for purely capillary waves. Using a series expansion similar to Stokes (1957), he wrote down an exact solution for capillary waves of arbitrary amplitude on an infinitely deep fluid and concluded a similar result was possible for finite depth. He found that for infinite depth, the wave of greatest height occurs when the distance from crest to trough is 0.73 wavelengths. A good overview of results on the computation of traveling wave solutions, including many results not discussed here, is found in the recent monograph by Vanden-Broeck [1]. Many of the results detailed there show the intricacies that follow from the inclusion of surface tension.

With solutions to Euler's equations on the periodic domain in hand, it is important to address their stability. Phillips (1960) [15] examined the dynamics of gravity waves on the surface of deep water and realized that when certain conditions are met, the waves behave as forced, resonant oscillators which cause energy transfer between the constituting wave trains. This work was supported by many experimental and numerical results such as the ones by Longuet-Higgins [16] and others. Phillips focused on a perturbation series expansion and the conditions necessary for the higher-order terms to satisfy the linear dispersion relation. He predicted that resonant triads are not possible for gravity waves in deep water. McGodrick showed that such triads are possible when surface tension is incorporated [15]. This was followed in 1967 by the works of Benjamin [4] and Whitham [5], who derived the criterion that Stokes waves on sufficiently deep water, i.e., $kh > 1.363$ with $k = 2\pi/L$, are modulationally unstable. For $kh < 1.363$, this instability is not present.

In 1968, these efforts were followed by the seminal work of Zakharov [17]. Starting from Euler's equations, he showed that the water wave problem is Hamiltonian. He wrote the energy in terms of the canonical variables $\eta(x,t)$ and $q(x,t) = \phi(x,\eta(x,t),t)$. Truncating in powers of the wave amplitude, he derived what is now called the Zakharov equation, from which the Nonlinear Schrödinger equation easily follows. This equation describes the dynamics of a modulationally unstable wave train and in this sense predicts what happens after the onset of the Benjamin–Feir instability.

By linearizing around a steady state solution, a stability eigenvalue problem is obtained whose spectrum determines the spectral stability of that solution. By examining the collision of eigenvalues in this spectrum, McLean (1982) [18] separated the instabilities in two classes. Building on the numerical work of Longuet-Higgins [19,20] and others, he obtained the maximal growth rates for different instabilities as a function of wave steepness. Exploiting the Hamiltonian nature of the problem [17], MacKay and Saffman (1986) [7] established necessary criteria for the onset of different instabilities as the amplitude of the solution is increased, within the framework of the general theory of Krein signatures [21]. These results are used below.

Many different ways of reformulating Euler's equations exist, mainly aimed at avoiding having to solve Laplace's equation in an unknown domain. The conformal mapping method is used to solve the one-dimensional water wave problem and leads to equations such as the ones used by Longuet-Higgins and Cokelet (1976) [16]. Another approach uses the canonical coordinates introduced by Zakharov [17] and defines a Dirichlet-to-Neumann operator; see Craig and Sulem (1993) [22]. Akers and Nicholls (2010) use the "Transformed Field Expansion" method, and they include the effects of surface tension [23]. Since we build on the work of Deconinck and Oliveras (2011) [3], the method most relevant to us is the reformulation due to Ablowitz, Fokas and Musslimani (2006) [24]. In this paper, the water wave problem is rewritten as two coupled equations, one local and one nonlocal. Since this is the basis for our work, this method is discussed below in some detail. The solutions of Deconinck and Oliveras [3] are in the form of a cosine series whose coefficients vary as the amplitude of the solution is increased. In [3], the effects of surface tension are