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The spectrum of finite depth water waves

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ABSTRACT

In this contribution we study the spectrum of periodic traveling gravity waves on a two-dimensional fluid of finite depth. We extend the stable and highly accurate method of Transformed Field Expansions to the finite depth case in the presence of both simple and repeated eigenvalues, and then numerically simulate the changes in the spectrum as the wave amplitude is increased. We also calculate explicitly the first non-zero correction to the flat-water spectrum, which we observe to accurately predict the stability (or instability) for all amplitudes within the disc of analyticity of the spectrum. In addition to computations of the spectrum, we also compute the radius of the disc of analyticity of the spectrum—the amplitude boundary beyond which neither the asymptotics nor the TFE method is applicable. We observe an instability which is analytically connected to the flat state for $kh \in (0.855, 1)$.

1. Introduction

The potential flow equations arise in a wide array of fluid mechanical problems, for instance, tsunami propagation, the motion of sandbars, and pollutant transport. Traveling wave solutions of these equations have the ability to propagate energy, momentum, and passive scalars (e.g., pollutants) around the world's oceans. In this study the spectral stability of such solutions under the influence of gravity in finite depth is considered.

This problem has a rich history of both numerical and asymptotic investigations, and the *Annual Review of Fluid Mechanics* is filled with articles summarizing various aspects of the field (see [1] for a particularly relevant and well-written example). The field has roots as early as Stokes, who first expanded periodic traveling water waves as a function of the wave slope in 1845 [2], an approach which has since become commonplace (see, e.g., [3–5]).

Regarding dynamic stability of these waveforms, real progress began in the 1960s with the discovery of the Benjamin–Feir instability [6] and, of particular relevance to the current study, the amplitude expansions which led to the development of the Resonant Interaction Theory (RIT) by Phillips [7] and Benney [8] (for an excellent review of the history of RIT see [9]). In RIT, the dynamics of the solution are predicted, asymptotically in the wave slope, by equations for the amplitudes of a small set of resonantly interacting frequencies, called triad or quartet equations (based on the number of frequencies in the interaction). For traveling water waves, RIT predicts the existence and growth rates of instabilities at frequencies which satisfy such interactions. Numerical studies have computed instabilities in the neighborhood of these resonances [10–13]. In the language of these numerical studies, the even interactions (quartets, sextets, etc.) are referred to as Class I instabilities while the odd interactions (triads, quintets, etc.) are referred to as Class II instabilities. We find that an eigenvalue's dependence on amplitude is characterized by the type of resonant interaction in which the eigenfunctions' frequencies take part.

To our knowledge, all stability studies to date concerning traveling wave solutions of the full water wave problem are numerical in nature. Further, almost all of these entail the linearization of the water wave equations about a *fixed* traveling wave solution followed by the numerical approximation of the resulting eigenvalue problem. Please see the classic results of [14,15] and the more recent computations of [16–18,13] for these "Direct Numerical Simulations" (DNS) of the spectral stability problem.

By contrast to the aforementioned DNS, the authors have embarked on an investigation of spectral stability using a rather different philosophy. In short, the spectrum of the water wave operator linearized about an analytic family of traveling waves is also analytic [19,20,5] (for simple eigenvalues) so that the eigenpair (λ, w) can be expanded in the strongly convergent Taylor series

$$\lambda = \lambda(\varepsilon) = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n, \qquad w = w(x; \varepsilon) = \sum_{n=0}^{\infty} w_n(x) \varepsilon^n,$$

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where ε is a wave height/slope parameter. These $\{\lambda_n, w_n\}$ have been approximated using the stable and highly (spectrally) accurate method [21] of "Transformed Field Expansions" (TFE) which was used to such great effect by one of the authors with F. Reitich [20,5] to simulate the underlying traveling waves. We refer the interested reader to [5] in particular for demonstrations of the capabilities of the TFE approach versus other Boundary Perturbation Methods including its favorable operation counts, lack of substantial numerical ill-conditioning, and applicability to *large* traveling wave profiles via numerical analytic continuation.

To put the present contribution into context we summarize our previous results:

- In [19] it was demonstrated that the spectrum of the water wave operator linearized about periodic traveling waves is *analytic* as a function of ε near simple eigenvalues.
- In [22] a TFE implementation of the theorem in [19] was used to numerically study the "evolution" of the spectrum for two-dimensional gravity waves in deep water. The role of singularities (in the Taylor series) in development of instability from the simple eigenvalue case was investigated.
- In [23] some conjectures regarding singularities in the spectrum and instability were resolved by comparing with a DNS of the spectrum in the gravity wave case.
- In [24] the TFE method was extended to include repeated eigenvalues and applied to deep-water gravity waves. RIT was used to find candidates for the "first" instabilities, those which arise at smallest wave slope.
- In [21] a rigorous numerical analysis of the TFE recursions was studied in a wide array of contexts, including the spectral stability problem.
- In [25] the TFE approach was extended to include the effect of surface tension. In deep water, triad instabilities were computed which are analytic in amplitude for fixed Bloch parameter.

In the present study we augment this line of results by:

- Extending the TFE method to the case of a fluid of finite depth.
- Computing *exactly* the first non-zero correction to the spectrum,

  $\lambda = \lambda_0 + \varepsilon^2\lambda_2 + \cdots.$
- Predicting the amplitude of instabilities and eigenvalue collisions using second order asymptotics.
- Estimating the radius of the disc of analyticity of the spectrum $\{\lambda, w\} = \{\lambda(\varepsilon), w(x;\varepsilon)\}$ from our numerical computations.

The ultimate result, the radius of the disc of analyticity, sets our method apart from direct numerical simulations of the spectrum. This radius highlights both the strengths and weaknesses of the approach. Boundary perturbation methods are limited in applicability by their radius of convergence; the TFE method cannot compute instabilities at amplitudes larger than this radius. Although the TFE method cannot compute these large amplitude instabilities, it does provide a mechanism for detecting their location, namely at the amplitudes and Bloch parameters at which the series loses analyticity [23]. The radius also gives an upper bound for the amplitude range over which asymptotic approximations, such as those presented here, may be expected to approximate the spectrum.

It is well known that small amplitude instabilities arise from collisions of flat state eigenvalues with opposite Krein signature [10]. Our method computes such instabilities as a series in amplitude with fixed Bloch parameter. Typical instabilities occur in bands of Bloch parameters whose width grows with amplitude [12,17]. We compute finite amplitude instabilities within these bands when the bands include the resonant Bloch parameters. We also compute the locations of these bands of Bloch parameters via the radius of convergence of our amplitude expansions. Based on our results, we conclude that in order for a boundary perturbation method to compute *all* instabilities at any finite amplitude, the method must allow the Bloch parameter to vary with amplitude. If not, only very special instabilities will be computable—those which occur at the same Bloch parameter at all amplitudes. An example of such an instability is presented for $kh \in (0.855, 1)$. Small amplitude asymptotics of the spectrum of the deep-water problem with amplitude-dependent Bloch parameters are calculated in [26]; these asymptotics are consistent with the conclusions presented here.

The paper is organized as follows: in Section 2 we introduce the water wave problem, followed by the TFE method for the spectral stability problem in Section 2.1, and the concept of Bloch periodicity (Section 2.2). In Section 2.3, we discuss the computation of the leading order spectrum, which we divide into cases based on the resonant character of the repeated eigenvalue. There are no triad interactions for two-dimensional water waves without surface tension, and we begin our discussion of degenerate quartet resonances in Section 2.4, followed by non-degenerate quartet resonances in Section 2.5. In Section 3 we present our numerical results including the radius of the disc of analyticity of the spectrum, as well as a computed instability. Conclusions and future areas of research are discussed in Section 4.

2. Spectral stability of traveling water waves

In this work we apply a perturbative approach to the spectral stability problem for water waves to compute the spectrum to all orders. The leading order correction to the flat state spectrum is exactly calculated and the general order correction is computed numerically. Two independent formulations are used for these computations. The exact leading order results are calculated in a classic Taylor expansion about the mean level, similar to the models in [27,28]. For computing the general order correction, classical Boundary Perturbation Methods have been observed to be numerically unstable in certain configurations. Consequently, we have selected the Transformed Field Expansions (TFE) approach [22–24] which executes a domain-flattening change of variables before expansion. This TFE method, justified analytically in [19] and, numerically, in [21], will be briefly presented in the following sections. Unlike the aforementioned classical Boundary Perturbation algorithms, the TFE method is both strongly convergent and numerically stable.

The widely accepted model for the motion of waves on the surface of a large body of water in the absence of surface tension or viscosity are the Euler equations

$$\phi_{xx} + \phi_{zz} = 0, \quad z < \varepsilon\eta, \tag{1a}$$

$$\phi_z = 0, \quad z = -H \tag{1b}$$

$$\eta_t + \varepsilon\eta_x\phi_x = \phi_z, \quad z = \varepsilon\eta, \tag{1c}$$

$$\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \phi_z^2\right) + \eta = 0, \quad z = \varepsilon\eta, \tag{1d}$$

where η is the free-surface displacement and ϕ is the velocity potential. These equations describe the motion of an inviscid incompressible fluid undergoing an irrotational motion. System (1) has been non-dimensionalized as in [27,24]. We assume that the wave slope, $\varepsilon = A/L$, is small (A is a typical amplitude and L, the characteristic horizontal length, is chosen in the non-dimensionalization so that the waves have spatial period 2π). Also, the vertical dimension has been non-dimensionalized using the wavelength, so the quantity H is non-dimensional ($H = kh$). As we will later use k_j for wavenumbers of eigenfunctions, we abandon the standard notation kh in favor of simply H.

In this work, we simulate the spectrum using the TFE approach of [20,22,24,25]. To describe the TFE approach, we recall the standard (see, e.g., [24]) truncation of the water wave domain to $\{-a <$