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# The gravity wave momentum flux in hydrostatic flow with directional shear over elliptical mountains

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ABSTRACT

Semi-analytical expressions for the momentum flux associated with orographic internal gravity waves, and closed analytical expressions for its divergence, are derived for inviscid, stationary, hydrostatic, directionally-sheared flow over mountains with an elliptical horizontal cross-section. These calculations, obtained using linear theory conjugated with a third-order WKB approximation, are valid for relatively slowly-varying, but otherwise generic wind profiles, and given in a form that is straightforward to implement in drag parametrization schemes. When normalized by the surface drag in the absence of shear, a quantity that is calculated routinely in existing drag parametrizations, the momentum flux becomes independent of the detailed shape of the orography. Unlike linear theory in the $Ri \to \infty$ limit, the present calculations account for shear-induced amplification or reduction of the surface drag, and partial absorption of the wave momentum flux at critical levels. Profiles of the normalized momentum fluxes obtained using this model and a linear numerical model without the WKB approximation are evaluated and compared for two idealized wind profiles with directional shear, for different Richardson numbers ($Ri$). Agreement is found to be excellent for the first wind profile (where one of the wind components varies linearly) down to $Ri = 0.5$, while not so satisfactory, but still showing a large improvement relative to the $Ri \to \infty$ limit, for the second wind profile (where the wind turns with height at a constant rate keeping a constant magnitude). These results are complementary, in the $Ri \gtrsim O(1)$ parameter range, to Broad's generalization of the Eliassen–Palm theorem to 3D flow. They should contribute to improve drag parametrizations used in global weather and climate prediction models.


## 1. Introduction

One of the many physical processes that are currently still unresolved in large-scale weather and climate prediction models is the effect of atmospheric gravity waves. These waves propagate in stratified fluids (such as the atmosphere typically is) [1], and are predominantly forced by flow over orography or convection occurring at horizontal scales (1–10 km) smaller than the grid spacings used operationally.

Waves generated by flow over mountains, which constitute a sizeable fraction of the total gravity waves, are known as mountain waves. These waves produce a surface drag on orography [2], whose reaction force decelerates the airflow, and must be parametrized to avoid substantial biases in the simulated global atmospheric circulation [3]. However, most well-known drag parametrizations are now outdated, having been developed in the 1990s [4,5]. The part of these parametrizations that accounts for the impact of wave propagation on the surface drag is based on linear wave theory, neglecting a number of important physical processes, such as non-hydrostatic effects, variations of the wind and static stability with height, and obviously wave nonlinearity, to mention just a few.

Linear theory is useful for developing drag parametrizations because it allows the drag to be expressed as a function of key orographic and incoming flow parameters. While a treatment of nonlinearity is, by definition, beyond its capabilities, wind profile effects can, in principle, be incorporated, although for generic wind profiles no analytical solutions exist, which precludes the derivation of simple drag expressions. Nevertheless, vertical wind shear has decisive implications for drag parametrization, since it may cause divergence of the wave momentum flux, which corresponds to a non-zero value of the reaction force exerted by the orography on the atmosphere [6,7].

Eliassen and Palm [8] demonstrated that the wave momentum flux in 2D flows is constant with height, even when the wind and static stability vary, except at levels where the wind vanishes

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(critical levels). This means that all the drag is exerted on the atmosphere at those particular discrete levels. More recently, however, Broad [9] showed that, in flows with directional shear over 3D mountains, critical levels (where the wind velocity is perpendicular to the horizontal wavenumber vector of a given spectral component of the waves) have a continuous distribution with height, and the variation of the wave momentum flux is coupled with the turning of the wind with height. Specifically, at a given level, the vertical derivative of the wave momentum flux vector (the momentum flux divergence) must be perpendicular to the wind velocity. This law, which is a generalization of the Eliassen–Palm theorem to 3D, places a strong constraint on the force exerted by mountains on the atmosphere.

However, the exact dependence of the drag on key flow parameters can only be determined by solving the corresponding mountain wave problem. Since this is not feasible analytically in the case of generic wind and static stability profiles, numerical or approximate methods must be employed. In the second category, one possibility is to split the atmosphere into a number of layers within which the wind velocity and static stability have a simple form (e.g., [10–13]), and then obtain the complete wave solutions and the corresponding drag. However, this approach lacks generality, and its results are often too cumbersome to implement in parametrizations. An alternative approach is to assume that the wind profile varies relatively slowly with height, and adopt a WKB approximation to obtain the wave solutions. Despite its limitations, this approach is considerably more general, being valid for generic (albeit slowly varying) wind profiles, and therefore providing a leading-order correction to the drag due to variation of the wind with height.

Teixeira et al. [14] calculated the surface drag using linear theory with a second-order WKB approximation for sheared, stationary, hydrostatic flow over an axisymmetric mountain and Teixeira and Miranda [15] did the same for 2D mountains. This model was extended to mountains with an elliptical horizontal cross-section by Teixeira and Miranda [16]. Subsequently, the wave momentum flux was calculated, using the same kind of approach, for flows with directional shear over an axisymmetric mountain [17]. This elucidated the filtering effect of critical levels with a continuous distribution with height in such flows, where the wave momentum flux may not be totally absorbed at relatively low Richardson numbers ($Ri$), but rather filtered. Teixeira and Miranda [17] did not calculate the wave momentum flux for flows with unidirectional shear or over 2D mountains, where critical levels occur at discrete heights (as mentioned above). These very particular cases had been addressed previously for simple wind profiles (without invoking the WKB approximation) by Booker and Bretherton [18] and Grubišić and Smolarkiewicz [19].

In the most important weather prediction and climate models, such as that running at the European Centre for Medium-Range Weather Forecasts (ECMWF), the Earth's orography is approximated in each model grid box as a mountain with an elliptical horizontal cross-section, with height, width and orientation calculated statistically from the real orography [20,4]. This approach appeals to a superposition principle (whereby the waves in each grid box do not interact with those in adjacent ones) whose strict validity is not straightforward, even for linearized flow. However, the fact that the width of these elliptical mountains is likely to be substantially smaller than the grid box (because they represent unresolved orography), and the assumption that the flow is hydrostatic (thus having very limited lateral wave propagation, especially at the surface), are consistent with the adopted, single-column approach.

In the present study we will develop and test the theory for calculating wave momentum fluxes in linearized, hydrostatic, non-rotating flow with directional shear over elliptical mountains. This theory provides nearly ready-to-use momentum flux and momentum flux divergence expressions, which may easily be incorporated into drag parametrizations. The results will be compared to those produced by a linear numerical model that does not assume the WKB approximation. Nonlinear effects were addressed previously in some detail for an axisymmetric mountain via comparisons with numerical simulations [17], and should not differ too much qualitatively for an elliptical geometry. Since it seems hopeless at present to formulate a physically self-consistent nonlinear mountain wave theory that is simple enough to implement in drag parametrizations, we neglect nonlinear effects altogether and focus here instead on evaluating the accuracy of the WKB approximation.

The remainder of this paper is organized as follows. Section 2 presents the linear mountain wave theory using the WKB approximation on which the subsequent momentum flux calculations are based. In Section 3, a linear numerical model that allows the treatment of arbitrary wind profiles (i.e. not assuming the WKB approximation) is briefly described. Section 4 explores the behaviour of the wave momentum flux with height for two representative idealized wind profiles. Finally, Section 5 summarizes the main conclusions of this study.

## 2. Linear WKB theory

The vertical flux of horizontal momentum associated with mountain waves forced by an arbitrarily-shaped isolated obstacle is defined here as

$$(M_x, M_y) = -\rho \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (u, v) w \, dxdy, \tag{1}$$

where $\rho$ is the density, $(u, v, w)$ is the velocity perturbation created by the waves, and $x$ and $y$ are the horizontal spatial coordinates. As in Teixeira and Miranda [17], this definition includes the minus sign, because the momentum flux is generally downward, and that convention makes it positive for a mean flow that is positive in the $x$ and/or $y$ direction.

Departing from linear theory with the Boussinesq approximation, assuming inviscid, non-rotating, stationary, hydrostatic flow and using additionally a WKB approximation to solve the Taylor–Goldstein equation (where the vertical wavenumber of the waves is expanded in a power series of a small parameter $\varepsilon$ proportional to $Ri^{-1/2}$ [14] up to third order), it can be shown that the momentum flux, correct up to second-order in $\varepsilon$, is given by

$$\begin{aligned}(M_x, M_y)(z) = {} & 4\pi^2 \rho_0 N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(k, l)|\hat{h}|^2}{(k^2 + l^2)^{1/2}} \\ & \times |U_0 k + V_0 l| \operatorname{sgn}(Uk + Vl) \left[1 - S(k, l, z)\right] \\ & \times e^{[S(k,l,z) - S(k,l,z=0)]} e^{-2\pi H(z - z_c) C(k,l)} \, dkdl \end{aligned} \tag{2}$$

(see Eqs. (27)–(29) of [17]), where

$$S(k, l, z) = \frac{1}{8} \frac{(U'k + V'l)^2}{N^2(k^2 + l^2)} + \frac{1}{4} \frac{(Uk + Vl)(U''k + V''l)}{N^2(k^2 + l^2)}, \tag{3}$$

$$C(k, l) = \frac{N(k^2 + l^2)^{1/2}}{|U_c'k + V_c'l|} \left[1 - \frac{1}{8} \frac{(U_c'k + V_c'l)^2}{N^2(k^2 + l^2)}\right]. \tag{4}$$

In the above equations $\rho_0$ is a reference density (assumed to be constant), $z$ is height, $N$ and $(U, V)$ are the Brunt–Väisälä frequency and velocity of the mean incoming flow, $(k, l)$ is the horizontal wavenumber vector of the waves, and $\hat{h}(k, l)$ is the Fourier transform of the terrain elevation $h(x, y)$. $z_c$ is the height of the critical level, the subscript 0 denotes values taken at the surface $z = 0$ and the subscript $c$ denotes values taken at the critical level, $H$ is the Heaviside step function and the primes denote differentiation with respect to $z$.