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Onset of convection in a porous layer with continuous periodic horizontal stratification, Part II: Three-dimensional convection

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ABSTRACT

The onset of convection in a porous layer which is heated from below is considered. In particular we seek to determine the effect of spatially periodic variations in the permeability field on the identity of the onset mode as a function of both the period P of this variation and its amplitude A. A Floquet theory is assumed in order to ensure that the analysis is as general as possible. It is found that convection is always three-dimensional and that the critical Rayleigh number always decreases as either the period or the amplitude of the permeability variation increases. Furthermore, the corresponding Floquet exponent ν is either 0 or 1, and the range of values of P over which $\nu = 1$ corresponds to the favoured mode has been obtained as a function of A.

1. Introduction

The onset of convective instability in a porous layer with a vertical temperature gradient has been the subject of very considerable attention particularly in the course of the last few decades. The first studies on this topic [1,2] were formulations of the classical Rayleigh–Bénard problem within the context of filtration processes in porous media as modelled through Darcy's law [3]. A development of these early studies on what might be called the Darcy–Bénard problem, was carried out by Palm et al. [4] in order to investigate the nonlinear effects under slightly supercritical conditions. These authors obtained an expression for the Nusselt number to high order in the supercritical parameter $(Ra - Ra_c)/Ra$, where Ra is the Rayleigh number and $Ra_c = 4\pi^2$ is its critical value at the onset of instability [3].

While there are many different extensions that one might apply to the Darcy–Bénard problems, some of which are the adoption of Brinkman and/or inertia effects, the dropping of the assumption of local thermal nonequilibrium, and the consideration of inclined layers or ones which conducting boundaries, the one which we concentrate on here is the effect of a heterogeneous permeability field. Heterogeneity could comprise layered materials or media where the permeability varies continuously with one or more coordinates, or else it could be random. McKibbin and O'Sullivan [5] studied a horizontally layered material and showed that large permeability differences are required for the multilayered medium to display onset conditions markedly different from those for a homogeneous layer. This analysis was developed further by Rees and Riley [6] by taking into account weakly nonlinear effects and they showed that double or multiple minimum loci for the Rayleigh number may exist at onset of instability. Studies of the Darcy–Bénard problem for heterogeneous porous media were carried out also by Nield and Simmons [7]. We mention that other sensible developments on this topic were achieved by McKibbin [8] and by Nield [9].

A situation where the permeability undergoes a periodic change was envisaged by De Wit and Homsy [10,11]. However, the kind of instability investigated by these authors is definitely different from the buoyancy-induced Rayleigh–Bénard instability. In fact, the physical effect leading to the instability is a concentration-dependent viscosity in the binary fluid saturating the porous medium. Much more closely related to the present paper is the work of Rees and Tyvand [12] (hereinafter referred to as Part 1) who considered a porous layer with a permeability which varies periodically in a horizontal direction. The analysis carried out in that paper was two-dimensional thus limiting the study to the behaviour of transverse rolls, i.e. ones with axes that are perpendicular to the direction of the x-axis along which the permeability changes periodically.

The aim of this contribution is to extend the investigation reported by Rees and Tyvand [12] from two-dimensional to three-dimensional modes. The Floquet theory, which was employed by

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Rees and Tyvand [12] to determine the selected two-dimensional modes of instability, is used in this study in order to determine whether two-dimensional modes or three-dimensional modes are favoured at onset of convection.

2. Governing equations

We consider a plane porous layer saturated by a Newtonian fluid. The thickness of the layer is H. The boundary planes at $z = 0$ and $z = H$ are impermeable and isothermal, and are held at the temperatures T_h and T_c, respectively, where $T_h > T_c$. The permeability, K, varies periodically in the x-direction and satisfies the following trigonometrical law,

$$K = K_0\left[1 + A\cos(\lambda x/H)\right], \tag{1}$$

where K_0 is the mean permeability, $A \in [0, 1)$ is a dimensionless amplitude, and λ is a dimensionless wavenumber which is such that $2\pi H/\lambda$ is the period of the permeability distribution (see Fig. 1).

The onset of convection in the porous layer is carried out under the following assumptions: (i) Darcy's law holds; (ii) the Oberbeck–Boussinesq approximation may be applied; (iii) the effective thermal conductivity and the effective volumetric heat capacity (the average product of the density and the specific heat) of the saturated porous medium are approximately uniform; (iv) there is local thermal equilibrium between the solid phase and the fluid phase; (v) no internal heating effect occurs. Assumption (iii) is a realistic description of porous media with an approximately uniform porosity. In spite of that, permeability can still be non-uniform. For instance, this may be the case with beds of particles or fibres, where the permeability may be inhomogeneous with a homogeneous porosity due to a variable morphology as, say, a spatially-varying average particle or fibre diameter. Another argument is that, while the effective thermal conductivity and the effective volumetric heat capacity depend on the porosity, permeability is controlled by the interconnected porosity of the medium. The latter parameter excludes from the evaluation of the void volume fraction the dead-end pores, where the fluid cannot actually flow.

We can express the governing equations in a dimensionless form by adopting the scalings,

$$\begin{aligned}
&\frac{1}{H}\,(x, y, z) \to (x, y, z)\,, \qquad \frac{\alpha_m}{\sigma H^2}\,t \to t,\\
&\frac{H}{\alpha_m}\,(u, v, w) \to (u, v, w)\,,\\
&\frac{T - T_c}{T_h - T_c} \to T, \qquad \frac{K_0 H}{\mu\alpha_m}\,\nabla p \to \nabla p.
\end{aligned} \tag{2}$$

Here, x, y, z and t denote the Cartesian coordinates and time, u, v, w are the velocity components, T is the temperature, ∇p is the dynamic pressure gradient, α_m is the effective thermal diffusivity of the saturated porous medium, μ is the fluid viscosity, and σ is the ratio between the effective volumetric heat capacity of the saturated porous medium and the volumetric heat capacity (the product of the density and the specific heat) of the fluid.

On account of Eq. (2), the dimensionless local balance equations for mass, momentum and heat transport may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3a}$$

$$\begin{aligned}
&u = -F(x)\,\frac{\partial p}{\partial x}, \qquad v = -F(x)\,\frac{\partial p}{\partial y},\\
&w = -F(x)\left(\frac{\partial p}{\partial z} - Ra\,T\right),
\end{aligned} \tag{3b}$$

$$\nabla^2 T = \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z}, \tag{3c}$$

while the boundary conditions are expressed as

$$\begin{aligned}
&z = 0: \qquad w = 0, \qquad T = 1, \qquad \frac{\partial p}{\partial z} = Ra,\\
&z = 1: \qquad w = 0, \qquad T = 0, \qquad \frac{\partial p}{\partial z} = 0.
\end{aligned} \tag{4}$$

Here, $F(x)$ and the Darcy–Rayleigh number Ra are defined respectively as,

$$F(x) = 1 + A\cos(\lambda x), \qquad Ra = \frac{\rho_c g\beta\,(T_h - T_c)\,K_0 H}{\mu\alpha_m}, \tag{5}$$

where ρ_c is the fluid density at the reference temperature T_c, g is the modulus of the gravitational acceleration $\mathbf{g}$, and β is the thermal expansion coefficient of the fluid.

The aim of this paper is to understand how the onset of three-dimensional convection depends on the values of the non-dimensional parameters, A, P and ν, where A is the amplitude of the permeability variation, $P = 2\pi/\lambda$ is the period of that variation, and ν is the Floquet exponent to be introduced below.

3. Basic solution and analysis of linear disturbances

A basic state which is a stationary solution of Eqs. (3) and (4) with a zero velocity exists and is given by

$$\begin{aligned}
&u_b = v_b = w_b = 0, \qquad T_b = 1 - z, \qquad \frac{\partial p_b}{\partial x} = 0,\\
&\frac{\partial p_b}{\partial y} = 0, \qquad \frac{\partial p_b}{\partial z} = Ra\,(1 - z)\,,
\end{aligned} \tag{6}$$

where the subscript b denotes the "basic solution". We introduce small-amplitude disturbances of the basic solution, Eq. (6), as follows,

$$\begin{aligned}
&(u, v, w) = (u_b, v_b, w_b) + \varepsilon\,(U, V, W)\,, \qquad T = T_b + \varepsilon\theta,\\
&\nabla p = \nabla p_b + \varepsilon\nabla\mathcal{P},
\end{aligned} \tag{7}$$

where ε is a perturbation parameter, such that $|\varepsilon| \ll 1$. We now substitute Eqs. (6) and (7) into Eqs. (3) and (4), and neglect terms which are of $O(\varepsilon^2)$. Thus, the system of linearised disturbance equations is given by

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \tag{8a}$$

$$\begin{aligned}
&U = -F(x)\,\frac{\partial\mathcal{P}}{\partial x}, \qquad V = -F(x)\,\frac{\partial\mathcal{P}}{\partial y},\\
&W = -F(x)\left(\frac{\partial\mathcal{P}}{\partial z} - Ra\,\theta\right),
\end{aligned} \tag{8b}$$

$$\nabla^2\theta = \frac{\partial\theta}{\partial t} - W, \tag{8c}$$

$$z = 0, 1: \qquad W = 0, \qquad \theta = 0. \tag{8d}$$

A pressure–temperature formulation is obtained by substituting Eq. (8b) into Eq. (8a), so that we finally obtain

$$\nabla^2\mathcal{P} = Ra\,\frac{\partial\theta}{\partial z} - G(x)\,\frac{\partial\mathcal{P}}{\partial x}, \tag{9a}$$

$$\nabla^2\theta = \frac{\partial\theta}{\partial t} + F(x)\left(\frac{\partial\mathcal{P}}{\partial z} - Ra\,\theta\right), \tag{9b}$$

$$z = 0, 1: \qquad \frac{\partial\mathcal{P}}{\partial z} = 0, \qquad \theta = 0, \tag{9c}$$

where

$$G(x) = \frac{F'(x)}{F(x)} = -\frac{A\lambda\sin(\lambda x)}{1 + A\cos(\lambda x)}, \tag{10}$$