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Mass transport in internal coastal Kelvin waves

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HIGHLIGHTS

- The Stokes and mean Eulerian drifts are trapped to the coast.
- The non-linear transports yield a jet-like flow along the southern Caspian coast.
- While the Stokes flux is zero, the Eulerian flux is not.
- The wave-induced drift may contribute to the mean circulation in the Caspian Sea.

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ABSTRACT

We investigate theoretically the mass transport in internal coastal Kelvin waves by integrating the horizontal momentum equations in the vertical. Applying a perturbation method, the time-averaged Lagrangian horizontal fluxes are determined to second order in wave steepness. The linear wave field is expanded in the vertical using orthogonal functions. Due to the orthogonality property of these functions, formulae for the non-linear Stokes drift and the mean vertically-averaged Eulerian transport driven by the radiation stress can be derived for arbitrary vertical variation of the Brunt–Väisälä frequency N. For values of N typical of the thermocline in the Caspian Sea, the calculation of the non-linear transports yields a jet-like mean flow along the coast, limited in the off-shore direction by the internal Rossby radius. It is suggested that this wave-induced mean drift may contribute to the mean circulation in the Caspian Sea.

1. Introduction

Most bodies of water at mid and lower latitudes have a pronounced vertical stratification. Together with the presence of a relatively straight coastline, this stratification may support internal Kelvin waves, being trapped in the region between the coast and the internal Rossby radius. For a small sea, like the Caspian Sea, with negligible tidal forcing, and a typical length scale of less than 1000 km, internal waves appear to be generated by temporal/spatial variations of the meso-scale wind field. In the southwestern part of the Caspian Sea, we find a strong thermocline around 60 m depth [1], which is much less than the total depth of the basin. To discuss the dynamics of such regions, a reduced-gravity model is often applied [2,3]. In this approach there are two layers of constant density, where the upper is thin and active, and the lower is very deep and passive with negligible velocity. Then, by replacing the acceleration due to gravity by the reduced gravity, the interfacial Kelvin wave can be obtained directly by analogy with the barotropic wave. However, the reduced-gravity model filters out higher baroclinic modes, and will not be used here. In addition, it yields an erroneous result for the Stokes drift, as shown in [4].

The main focus of the present study is the mean drift induced by internal coastal Kelvin waves. Such waves possess mean momentum, and hence induce a Stokes drift. In addition, since the waves are damped due to friction, they will generate a mean Eulerian flow. For spatially damped waves, this baroclinic flow is driven by the radiation stress in the waves. This theme has been thoroughly discussed for barotropic flows by Longuet-Higgins and Stewart [5]. It has also been studied for interfacial coastal Kelvin waves in a reduced gravity context [6]. We here consider this effect in the continuously stratified case. This application appears to be novel.

The rest of this paper is organized as follows: in the Section 2 we state the basic assumptions and the governing equations, while in the Appendix we consider linear internal coastal Kelvin waves. In Section 3 we derive the Stokes drift in internal Kelvin waves, and in Section 4 we apply the results to the Caspian Sea. The mean vertically-averaged Eulerian velocity is derived in Section 5, while Section 6 contains a discussion and some concluding remarks.

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2. Basic assumptions and governing equations

We consider a viscous ocean of constant depth H, and we chose a Cartesian coordinate system (x, y, z) such that the origin is situated at the undisturbed sea surface, the x-axis is directed along the coast, the y-axis is positive towards the sea, and the z-axis is directed vertically upwards. The respective unit vectors are $(\vec{i}, \vec{j}, \vec{k})$. The reference system rotates about the vertical axis with angular velocity $(f/2)$, where f is the constant Coriolis parameter. Furthermore, we use an Eulerian description of motion, which means that all dependent variables are functions of x, y, z and time t. We take that the horizontal scale of the motion is so large compared to the depth that we can make the hydrostatic approximation in the vertical. Furthermore, we apply the Boussinesq approximation for the density ρ. We also take that the density of an individual fluid particle is conserved. The governing equations for this problem then become

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \nabla \vec{v}_h = -f\vec{k} \times \vec{v}_h - \frac{1}{\rho_r}\nabla_h p + \frac{\partial}{\partial z}\left[\frac{\vec{\tau}_h}{\rho_r}\right], \tag{1}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0, \tag{3}$$

$$\nabla \cdot \vec{v} = 0. \tag{4}$$

Here $\vec{v} = (u, v, w)$ is the velocity vector, p is the pressure, subscript h means horizontal values, and $\vec{\tau}_h = (\tau^{(x)}, \tau^{(y)})$ is the turbulent stress in the horizontal direction. Furthermore, ρ_r is a constant reference density, and g the acceleration due to gravity. We take that there is no forcing from the atmosphere in this problem, i.e. at the surface: $\tau_s^{(x)} = \tau_s^{(y)} = p_s = 0$. The surface is material and given by $z = \eta(x, y, t)$. Integrating our governing equations from the horizontal bottom to the moving surface, we obtain equations for the horizontal Lagrangian volume transport (U_L, V_L) in the fluid:

$$\frac{\partial U_L}{\partial t} - fV_L = -\frac{1}{\rho_r}\frac{\partial}{\partial x}\int_{-H}^{\eta} p dz - \frac{\partial}{\partial x}\int_{-H}^{\eta} u^2 dz - \frac{\partial}{\partial y}\int_{-H}^{\eta} uv dz - \frac{\tau_B^{(x)}}{\rho_r}, \tag{5}$$

$$\frac{\partial V_L}{\partial t} + fU_L = -\frac{1}{\rho_r}\frac{\partial}{\partial y}\int_{-H}^{\eta} p dz - \frac{\partial}{\partial x}\int_{-H}^{\eta} uv dz - \frac{\partial}{\partial y}\int_{-H}^{\eta} v^2 dz - \frac{\tau_B^{(y)}}{\rho_r}, \tag{6}$$

where we have defined

$$U_L = \int_{-H}^{\eta} u dz, \qquad V_L = \int_{-H}^{\eta} v dz. \tag{7}$$

Furthermore, $(\tau_B^{(x)}, \tau_B^{(y)})$ are the turbulent bottom stresses.

In principle we expand our solutions in series after the wave steepness as a small parameter (although we retain our dimensional variables). The first order (linear) wave solution will be marked by a tilde, while to second order we consider averages over the wave period. Such (non-linear) quantities will be marked by an over-bar.

We consider trapped internal waves propagating along the x-axis. The waves result from small perturbations from a state of rest characterized by a horizontally-uniform stable stratification $\rho_0(z)$. We take that the velocity in the y-direction vanishes identically, characterizing the Kelvin wave. Introducing the vertical displacement $\xi(x, y, z, t)$ of the isopycnals from their original horizontal position, linear theory yields $\partial\tilde{\xi}/\partial t = \tilde{w}$, where the tilde is used to mark linear perturbation quantities. The linearized system of equations for internal coastal Kelvin waves can then be written from (1) to (4):

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= -\frac{1}{\rho_r}\frac{\partial \tilde{p}}{\partial x} + \frac{\partial}{\partial z}\left[\frac{\tilde{\tau}^{(x)}}{\rho_r}\right], \\ f\tilde{u} &= -\frac{1}{\rho_r}\frac{\partial \tilde{p}}{\partial y}, \\ \frac{\partial \tilde{p}}{\partial z} &= -\rho_r N^2 \tilde{\xi}, \\ \frac{\partial \tilde{u}}{\partial x} &= -\frac{\partial^2 \tilde{\xi}}{\partial z \partial t}. \end{aligned} \tag{8}$$

Here N is the Brunt–Väisälä frequency defined by

$$N^2 = -\frac{g}{\rho_r}\frac{d\rho_0(z)}{dz}. \tag{9}$$

The variables may be separated into normal modes [7], and we refer to [8] for details. For didactic reasons we give a short account of the wave solutions in the Appendix.

In summary, letting real parts represent the physical solution, we have from the Appendix:

$$\tilde{\xi} = \sum_{n=1}^{\infty} \xi_n(x, y, t)\phi_n(z), \qquad \tilde{u} = \sum_{n=1}^{\infty} u_n(x, y, t)\phi_n'(z), \tag{10}$$

where ϕ_n is given by (A.2), and

$$\xi_n = A_n \exp\left(-\alpha_n x - a_n^{-1} y\right)\cos\left(k_n x + l_n y - \omega t\right), \tag{11}$$

$$u_n = c_n A_n \exp\left(-\alpha_n x - a_n^{-1} y\right)\left[\cos\left(k_n x + l_n y - \omega t\right) + \frac{\alpha_n}{k_n}\sin\left(k_n x + l_n y - \omega t\right)\right].$$

As for temporally damped waves [9], we note that the lines of constant phase for spatially damped coastal Kelvin waves are straight lines slanting backwards. In principle, the displacement amplitudes $A_1, A_2, A_3 \ldots$ must be determined from field observations, or from analytical/numerical model runs with appropriate forcing.

3. The Stokes drift

As first shown by Stokes [10], periodic waves possess non-zero mean wave momentum, leading to a net drift of particles in the fluid. This mean drift is referred to as the Stokes drift, and is basically related to the inviscid part of the wave field, eventually modified by a slow temporal or spatial viscous decay of wave amplitude. To second order in wave steepness the Stokes drift in the x-direction can be expressed by the Eulerian wave field [11]:

$$\bar{u}_s = \overline{\left(\int \tilde{u} dt\right)\frac{\partial \tilde{u}}{\partial x}} + \overline{\left(\int \tilde{v} dt\right)\frac{\partial \tilde{u}}{\partial y}} + \overline{\left(\int \tilde{w} dt\right)\frac{\partial \tilde{u}}{\partial z}}, \tag{12}$$

where the over-bar denotes average over one wave period $T = 2\pi/\omega$. In the present problem $\tilde{v} = 0$, and $\tilde{w} = \tilde{\xi}_t$. Hence, from (10) and (11) for internal Kelvin waves:

$$\bar{u}_s = \frac{1}{2}\sum_{n=1}^{\infty} c_n A_n^2 \left(\left(\phi_n'\right)^2 + \phi_n \phi_n''\right)\exp\left(-2\alpha_n x - 2a_n^{-1} y\right). \tag{13}$$

The expression (13) is valid for arbitrary $N(z)$. Inserting from (A.2), we obtain that

$$\bar{u}_s = \frac{1}{2}\sum_{n=1}^{\infty} c_n A_n^2 \left(\left(\phi_n'\right)^2 - \frac{N^2}{c_n^2}\phi_n^2\right)\exp\left(-2\alpha_n x - 2a_n^{-1} y\right). \tag{14}$$