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Deformation of an elastic substrate due to a sessile drop

C. Pozrikidis*, A.I. Hill

Department of Chemical Engineering, University of Massachusetts, Amherst, MA 01003, USA

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ABSTRACT

The small deformation of a horizontal elastic substrate due to an axisymmetric sessile drop resting on the substrate is calculated for exact hydrostatic interfacial shapes computed by solving the Young–Laplace equation using an iterative method. The drop height, base radius, and top interfacial curvature are documented in terms of the ratio of the capillary length to the equivalent drop radius defined with respect to the drop volume. The substrate is subjected to a uniform discoidal load over the drop base due to the elevated interior pressure, and a singular ring load with a normal and possibly a tangential component around the contact line. The relative contributions of the Laplace pressure and drop weight to the normal load over the drop base are delineated in terms of the drop size. Once the interfacial shapes are available, the substrate deformation field is reconstructed from three modular kernels originating from the axisymmetric versions of the Boussinesq and Cerruti solutions for a semi-infinite elastic medium. The effect of gravity causing the drop to deviate from the spherical shape has an important influence on the magnitude and profile of the substrate deformation. Numerical results illustrate the effect of spreading of the vertical component of the capillary force and the significance of the tangential component of the capillary force around the contact line.

1. Introduction

A simple interface between two immiscible fluids imparts to a three-phase contact line supporting the interface a capillary force that is tangential to the interface and normal to the contact line. A structured interface, such as a membrane enclosing a vesicle or biological cell, may impart a tangential force with arbitrary orientation determined by the mechanical response of the interface regarded as a thin shell. In the case of a simple interface, Young's equation provides us with an expression for the contact angle at each point along the contact line in terms of the interfacial tension and two corresponding tangential tensions specific to the two solid–fluid pairs. Conversely, the contact angle is established to ensure the satisfaction of the tangential force balance. However, the validity of Young's equation in its primary form has been questioned by several authors (e.g., [1]).

The satisfaction of the normal force balance at the contact line has been the subject of extensive discussion and speculation aimed at providing a sensible physical explanation (e.g., [2–7]). A consensus appears to have been reached that the normal component of the capillary force is counteracted by elastic stresses developing due to substrate deformation. The surface displacement is small for stiff solid surfaces, but may be significant for highly deformable substances and gels. When the substrate deformation is small, the nominal contact angle changes by a small amount that can be estimated on the basis of energetics, taking into account the work necessary to deform the elastic solid. A highly deformable substrate, such as a free elastic sheet, is unable to withstand the capillary force and may wrap around or even encapsulate the interface. Strong elasto-capillary interactions are responsible for highly deformed configurations involving elastic beams and flexible sheets (e.g., [8–10]).

A sessile drop resting on a horizontal plane provides us with a convenient physical and conceptual prototype for studying substrate deformation due to a capillary force, as recently reviewed [11]. Theoretical studies of substrate deformation and its influence on the contact angle have been conducted on several occasions following Lester's [2] analysis for drops resting on thick blocks and thin plates. Experimental observations have been carried out and illuminating photographs have been published in recent years (e.g., [11–14]). Notable is the theoretical work of Fortes [15] and a tetralogy of papers by Rusanov [2–5] who derived expressions for the surface deformation field of a semi-infinite medium and proceeded to estimate consequent changes in the apparent contact angle. Rusanov's original derivation of the surface displacement field was repeated and rederived in identical or alternative forms by slightly different methods by subsequent authors. Molecular rearrangements near the contact line play an important role.

One way of preventing the unphysical logarithmic singularity of the surface displacement field at the contact line is to spread the capillary force over a strip with molecular or higher dimensions.

* Corresponding author. Tel.: +1 413 577 2157.
E-mail address: cpozrikidis@ecs.umass.edu (C. Pozrikidis).

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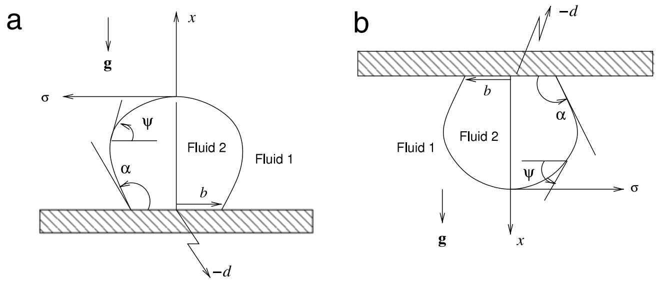

Fig. 1. Illustration of (a) an axisymmetric sessile liquid drop resting on a horizontal plane, and (b) an axisymmetric pendant liquid drop hanging underneath a horizontal plane.

When this is done, a scaling argument can be used to show that the maximum vertical displacement of the contact line is on the order $\gamma \sin\alpha/E$, where γ is the interfacial tension, E is the modulus of elasticity of the substrate, and α is the contact angle. Another way of suppressing the logarithmic singularity is to argue that plastic stresses developing near the contact line restrain the infinite deformation.

Computations by previous authors of the surface displacement field due to a sessile drop resting on a deformable substrate have neglected gravitational effects and assumed that the drop has a spherical shape (e.g., [11]). Although the spherical shape is accurate when the drop size is much smaller than the capillary length, it is only an approximation for large drops. Our main goal in this paper is to illustrate the effect of drop flattening due to significant gravitational effects on the substrate deformation. In the first stage, hydrostatic drop shapes are computed by solving the Young–Laplace equation for axisymmetric configurations using a conventional shooting method. In the second stage, the surface deformation field is calculated using the axisymmetric versions of the Boussinesq and Cerruti solutions for the deformation of a semi-infinite solid under the action of a solitary surface load. The pertinent surface deformation kernels are presented in convenient modular forms. The hydrostatic shapes are discussed in Section 2 where Rusanov's asymptotic analysis for flat drops is presented and validated by the results of the numerical computations. The surface displacement is discussed in Section 3, and the results are summarized in Section 4. It should be emphasized that the theoretical formulation presented in this work and the numerical results are restricted to small surface deformations corresponding to stiff elastic substrates.

2. Shapes of sessile and pendant drops

Consider an axisymmetric drop of a fluid labeled 2 resting above or hanging underneath a horizontal plane. The drop is surrounded by an ambient fluid labeled 1, as illustrated in Fig. 1. The resting drop shown in Fig. 1(a) is a *sessile* drop, and the hanging drop shown in Fig. 1(b) is a *pendant* drop. Our analysis also applies for a gas bubble, regarded as a zero-density drop, $\rho_2 = 0$. Our objective is to compute the shape of the interface for specified surface tension, γ, contact angle, α, and drop volume, V_D.

The pressure distribution in the two fluids is given by

$$p^{(1)}(x) = -s_1 \rho_1 g x + \pi_1, \qquad p^{(2)}(x) = -s_1 \rho_2 g x + \pi_2, \tag{2.1}$$

where π_1 and π_2 are two reference pressures. The coefficient s_1 is equal to 1 for a sessile drop and -1 for a pendant drop, reflecting the orientation of gravity with respect to the positive direction of the x axis. The shape of the interface is governed by the Laplace–Young equation,

$$2\,\kappa_m = -s_1 \frac{\Delta\rho\, g}{\gamma} x + B, \tag{2.2}$$

where $\Delta\rho = \rho_2 - \rho_1$ and $B \equiv (\pi_2 - \pi_1)/\gamma$ is an *a priori* constant with dimensions of inverse length. In terms of the square of the capillary length, $\ell^2 \equiv \gamma/(|\Delta\rho|\, g)$, Eq. (2.2) takes the compact form

$$2\,\kappa_m = -s_1 s_2 \frac{x}{\ell^2} + B, \tag{2.3}$$

where the coefficient s_2 is equal to 1 when $\rho_2 > \rho_1$ or -1 when $\rho_2 < \rho_1$.

Applying (2.2) at the origin, we find that the constant B is equal to twice the mean curvature of the interface at the centerline,

$$B = 2\,\kappa_m^0. \tag{2.4}$$

If the shape of the drop surface is described by the function

$$\sigma = w(x), \tag{2.5}$$

then the mean curvature of the interface is given by the expressions

$$2\,\kappa_m = -\frac{w''}{(1+w'^2)^{3/2}} + \frac{1}{w}\frac{1}{\sqrt{1+w'^2}} = -\left(\frac{w'}{\sqrt{1+w'^2}}\right)' + \frac{1}{w}\frac{1}{\sqrt{1+w'^2}} \tag{2.6}$$

and

$$2\,\kappa_m = \frac{1}{w w'}\left(\frac{w}{\sqrt{1+w'^2}}\right)', \tag{2.7}$$

where a prime denotes a derivative with respect to x (e.g., [16]).

Substituting the second expression for the mean curvature given in (2.6) into the Young–Laplace equation (2.3), we obtain

$$-\left(\frac{w'}{\sqrt{1+w'^2}}\right)' + \frac{1}{w}\frac{1}{\sqrt{1+w'^2}} = -s_1 s_2 \frac{x}{\ell^2} + B. \tag{2.8}$$

Integrating with respect to x across the height of the drop, from $x = -d$ to 0, we obtain

$$1 - \cos\alpha = 2\,\sin^2\frac{\alpha}{2} = s_1 s_2 \frac{1}{2}\frac{d^2}{\ell^2} + Bd - J, \tag{2.9}$$

where

$$J \equiv \int_{-d}^{0} \frac{1}{w}\frac{\mathrm{d}x}{\sqrt{1+w'^2}} = \int_{-d}^{0} \frac{\sin\psi}{\sigma}\,\mathrm{d}x, \tag{2.10}$$

and the slope angle ψ is defined by the equation $w' = -\cot\psi$, as shown in Fig. 1. The presence of the dimensionless integral J associated with the second principal curvature distinguishes the axisymmetric from a two-dimensional drop. To compute this integral, the interfacial shape must be known.

When $s_1 s_2 = 1$, and the top of the drop is nearly flat due to strong gravitational effects, $Bd \simeq 0$. Rusanov [6] noted that, under these conditions, the integral J can be approximated by integrating around the sides of the drop where $w \simeq b$, where b is the drop base radius, as shown in Fig. 1. Using an analytical expression for the profile of a two-dimensional meniscus (e.g., [17]), we obtain

$$w' \simeq \frac{1}{\phi}\frac{2-\phi^2}{\sqrt{4-\phi^2}}, \tag{2.11}$$

and approximate

$$J \simeq \frac{\ell}{2\,b}\int_0^{d/\ell} \phi\,\sqrt{4-\phi^2}\,\mathrm{d}\phi = \frac{1}{6}\frac{\ell}{b}\left[8 - \left(4 - \frac{d^2}{\ell^2}\right)^{3/2}\right], \tag{2.12}$$

where $\phi = \tilde{x}/\ell$ and $\tilde{x} \equiv x + d$ is the distance of the interface from the support. Eq. (2.12) is consistent with Rusanov's [6, Eq. (5)]