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Existence and nonexistence of solutions on opposing mixed convection problems in boundary layer theory☆

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ABSTRACT

We introduce an integral equation to study the opposing mixed convection problems in boundary layer theory. This equation is of singularities and two integrands take negative values. By means of some special analytical techniques, we prove the existence and the nonexistence of positive solutions of this equation and utilize it to treat analytically the mixed convection parameter $\varepsilon < -1$ and the temperature parameter $\lambda > 0$ involved in the problems mentioned above. Previous results only treated the case $\lambda = 0$ or $\varepsilon \geq -1$.

1. Introduction

In this research, we are concerned with the existence and the nonexistence of analytical solutions of opposing mixed convection for the following third-order nonlinear differential equation

$$f'''(\eta) + (1+\lambda)f(\eta)f''(\eta) + 2\lambda(1-f'(\eta))f'(\eta) = 0 \quad \text{on } [0,\infty) \tag{1.1}$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 1+\varepsilon \quad \text{and} \quad f'(\infty) = 1, \tag{1.2}$$

which has been used to describe the plane mixed convection boundary layer flow near a semi-infinite vertical plate embedded in a saturated porous media, with a prescribed power law of the distance from the leading edge for the temperature, where λ is the parameter of the temperature profile power-law and ε is the mixed convection parameter, namely, $\varepsilon = \frac{R_a}{P_e}$ with R_a the *Rayleigh* number and P_e the *Péclet* number. For $\varepsilon = 0$, (1.1)–(1.2) corresponds to the forced convection, for $\varepsilon > 0$, it corresponds to aiding the mixed convection and for $\varepsilon < 0$, it corresponds to the opposing mixed convection. For more details on the physical derivation and the numerical treatments of (1.1)–(1.2), see [1,2].

Regarding the study of (1.1)–(1.2), Guedda [3] studied the existence of infinitely many solutions of (1.1)–(1.2) for $-1 < \lambda < 0$ and $-1 < \varepsilon < \frac{1}{2}$ and the nonexistence of nonnegative solutions for $\lambda \leq -1$ and $\varepsilon \geq \frac{1}{2}$; Brighi and Hoernel [4] proved the existence and the uniqueness of convex and concave solutions of (1.1)–(1.2) for $\lambda > 0$, $-1 < \varepsilon < 0$ and $\varepsilon > 0$. For $\lambda = 0$, it is well known that (1.1)–(1.2) is the Blasius equation, Hussaini and Lakin [5] showed that there exists $\varepsilon_c < -1$ such that (1.1)–(1.2) has a solution for $\varepsilon \geq \varepsilon_c$ and no solution for $\varepsilon < \varepsilon_c$. Numerical result showed $\varepsilon_c \doteq -1.3545$. For further results on the Blasius equation, one may refer to [6,7].

However, to our knowledge, for $\lambda \neq 0$ and $\varepsilon < -1$, there exists little analytic study on the existence and the nonexistence of analytical solutions of (1.1)–(1.2).

In this research, our interest is focused on this case. We shall establish the relation between (1.1)–(1.2) and an integral equation (see (2.3)) and prove results on the existence and the nonexistence of positive solutions of this equation, which derive our desired conclusions as follows.

Theorem 1.1. *There exist $\varepsilon_* \in (-1.807, -1.806)$ and $\varepsilon^* \in (-1.193, -1.192)$ such that*

(i) *(1.1)–(1.2) has no convex solutions for any $\lambda > 0$ and each $\varepsilon \leq \varepsilon_*$.*

(ii) *(1.1)–(1.2) has a convex solution for each $\lambda > 0$ and each $\varepsilon \in [\varepsilon^*, -1)$.*

It is well known that analytical and numerical studies of similarity solutions are both of considerable practical importance in many fields and can provide an important standard of comparison without introducing the complication of non-similar solutions, much attention is always focused on them. One may refer to, for example, the review and extension of similarity solutions [8,9] and some

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recent studies such as boundary layer flows [10–13], magnetohydrodynamic (MHD) [12,14,15], etc. and the references therein.

2. Relation between (1.1)–(1.2) and an integral equation

Let $f(\eta) \in C^3(\mathbb{R}^+)$, $f(\eta)$ is said to be a convex solution if $f(\eta)$ satisfies (1.1)–(1.2) and $f''(\eta) > 0$ for $\eta \in \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$.

Throughout this paper, we assume $\lambda > 0$ and $\varepsilon < -1$. For the sake of convenience, let

Notation. $\beta = 1 + \varepsilon, \varphi(\lambda) = \frac{1+3\lambda}{1+\lambda}$.

We change (1.1) and (1.2) into (2.1) and (2.2), which can be found in the literature (see, e.g., Remark 1.2 in [10]).

Proposition 2.1. *The problem (1.1)–(1.2) is equivalent to the following boundary value problem*

$$F''' + FF'' + (\varphi(\lambda) - 1)(1 - F')F' = 0 \quad \text{for } \eta \in \mathbb{R}^+ \tag{2.1}$$

subject to the boundary conditions

$$F(0) = 0, \qquad F'(0) = \beta \quad \text{and} \quad F'(\infty) = 1, \tag{2.2}$$

where $F(\eta) = \sqrt{1+\lambda} f(\frac{\eta}{\sqrt{1+\lambda}})$.

Let

$$\Gamma = \{F \in C^3(\mathbb{R}^+) : F''(\eta) > 0 \text{ on } \mathbb{R}^+\}$$

and

$$Q = \{z \in C[\beta, 1] : z(t) > 0 \text{ for } t \in [\beta, 1)\},$$

where $C[\beta, 1]$ denotes the continuous functions space with the maximum norm $\|z\| = \max\{|z(t)| : t \in [\beta, 1]\}$.

Now, we establish the relation between (2.1)–(2.2) and an integral equation.

Lemma 2.1. *If $F \in \Gamma$ satisfies (2.1)–(2.2), then*

$$z(t) = \varphi(\lambda) Az(t) + (1 - t)Bz(t) \quad \text{for } t \in [\beta, 1] \tag{2.3}$$

has a solution $z \in Q$, where

$$Az(t) = \int_t^1 \frac{s(1-s)}{z(s)} ds \quad \text{and} \quad Bz(t) = \int_\beta^t \frac{s}{z(s)} ds.$$

Proof. Assume that $F \in \Gamma$ satisfies (2.1)–(2.2). As the same as the proof of Proposition 3.1 (4) in [10], we have $F''(\infty) = 0$.

Let $t = F'(\eta)$ be a dependent variable, $z(t) = F''(\eta)$. Since F' is strictly increasing on $[0, \infty)$ by $F''(\eta) > 0$ on $\mathbb{R}^+$, then $\beta \le t < 1$, $z(t) > 0$ on $[\beta, 1)$, $z(1) = F''(\infty) = 0$ and $z \in Q$. By $1 = F''(\eta)\frac{d\eta}{dt}$, we see

$$\frac{d\eta}{dt} = \frac{1}{F''(\eta)} = \frac{1}{z(t)}, \qquad F(\eta) = \int_0^\eta F'(\sigma)d\sigma = \int_\beta^t \frac{s}{z(s)} ds$$

by setting $s = F'(\sigma)$ and $F'''(\eta) = z'(t)\frac{dt}{d\eta} = z(t)z'(t)$.

Substituting $F(\eta), F'(\eta), F''(\eta)$ and $F'''(\eta)$ into (2.1) implies

$$z'(t) = \frac{(1 - \varphi(\lambda))t(1-t)}{z(t)} - Bz(t) \quad \text{for } t \in [\beta, 1) \tag{2.4}$$

and $z(1) = 0$.

Since

$$\frac{(1 - \varphi(\lambda))t(1-t)}{z(t)} - Bz(t) = \left[\frac{(1 - \varphi(\lambda))t(1-t)}{z(t)} - \int_0^t \frac{s}{z(s)} ds\right] - Bz(0), \tag{2.5}$$

integrating (2.4) from t to 1, using the fact that both terms in the bracket of the right hand side in (2.5) have the same sign for $t \in [0, 1]$ and noticing that

$$\begin{aligned} &\int_t^1 \int_\beta^s \frac{\mu}{z(\mu)} d\mu ds \\ &= \int_\beta^t \left[\int_t^1 \frac{\mu}{z(\mu)} ds\right] d\mu + \int_t^1 \left[\int_\mu^1 \frac{\mu}{z(\mu)} ds\right] d\mu \\ &= (1-t)Bz(t) + \int_t^1 \frac{s(1-s)}{z(s)} ds, \end{aligned} \tag{2.6}$$

we see that $z(t)$ satisfies (2.3). □

Since (2.3) contains the improper integrals $Az(t)$ and $Bz(t)$, we first investigate some properties of solutions of (2.3).

Let $z \in Q$ be a solution of (2.3); then

$$z(1) = 0 \quad \text{and} \quad \lim_{t \to 1^-} (1-t)Bz(t) = 0.$$

In fact, if $z(1) > 0$, then $z(t) > 0$ for $t \in [\beta, 1]$. This implies that two integrands in (2.3) are continuous and then $z(1) = 0$, a contradiction. Since $Az(t)$ and $(1-t)Bz(t)$ are well-defined on $[\beta, 1]$ and $Az(1) = 0$, we immediately get that $\lim_{t\to 1^-}(1-t)Bz(t) = 0$. Proposition 2.3(ii) will show that $\lim_{t\to 1^-}(1-t)Bz(t)$ is an indeterminate form of type $0 \times \infty$.

The following proposition shows the equivalence between (2.3) and a first-order differential equation with suitable boundary condition.

Proposition 2.2. *Let $z \in Q$; then z is a solution of (2.3) if and only if $z(1) = 0$ and*

$$z'(t) = \frac{(1 - \varphi(\lambda))t(1-t)}{z(t)} - Bz(t) \quad \text{for } \beta \le t < 1. \tag{2.7}$$

Proof. Assume that $z \in Q$ is a solution of (2.3). Differentiating (2.3) with t and combining $z(1) = 0$, we know that (2.7) holds. Conversely, integrating (2.7) from t to 1 and using the same arguments used in the proof of Lemma 2.1 and (2.6), we know that z is a solution of (2.3). □

Proposition 2.3. *Let $z \in Q$ be a solution of (2.3); then*

(i) $z(t) \ge z(0)(1-t) := \gamma(t)$ for $t \in [0, 1]$.

(ii) $\int_\beta^1 \frac{1}{z(s)} ds = \infty$.

Proof. Assume that $z \in Q$ is a solution of (2.3).

(i) If there exists $t \in [0, 1]$ such that $z(t) < \gamma(t)$, then $t \in (0, 1)$ since $z(0) = \gamma(0)$ and $z(1) = 0 = \gamma(1)$. Let $\psi(t) = z(t) - \gamma(t)$ and $\xi \in (0, 1)$ satisfying

$$\psi(\xi) = \min\{\psi(t) : t \in [0, 1]\} < 0.$$

Then $\psi'(\xi) = 0$, $\psi''(\xi) \ge 0$ by the second derivative test and $z'(\xi) = \gamma'(\xi) = -z(0)$ and then $z''(\xi) = \psi''(\xi) \ge 0$.

Differentiating (2.7) with t, we have

$$z''(t) = \frac{(1 - \varphi(\lambda))(1 - 2t) - t}{z(t)} + \frac{(\varphi(\lambda) - 1)(1-t)tz'(t)}{z^2(t)}$$

for $0 \le t < 1$.

Since $z(\xi) < \gamma(\xi) = z(0)(1 - \xi)$, we see

$$\frac{(1-\xi)\xi z'(\xi)}{z^2(\xi)} = -\frac{\xi}{z(\xi)} \frac{(1-\xi)z(0)}{z(\xi)} < -\frac{\xi}{z(\xi)}.$$

This, together with $\varphi(\lambda) - 1 = \frac{2\lambda}{1+\lambda} > 0$, implies

$$\begin{aligned} z''(\xi) &< \frac{(1 - \varphi(\lambda))(1 - 2\xi) - \xi}{z(\xi)} - \frac{(\varphi(\lambda) - 1)\xi}{z(\xi)} \\ &= \frac{(1 - \varphi(\lambda))(1 - \xi) - \xi}{z(\xi)} < 0. \end{aligned}$$

This is a contradiction. Hence, (i) holds.