European Journal of Mechanics B/Fluids 43 (2014) 185–190

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Free wave modes in elliptic cylindrical containers

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HIGHLIGHTS

- Gravity–capillary waves in cylindrical containers with elliptical cross-section are studied.
- We show the dependence of the natural frequencies and the frequency shift on the eccentricity e.
- We retrieve the well-known case of a circular basin for $e = 0$.
- The elliptical confinement can be used to modulate the damping ratio.

ARTICLE INFO

Article history:
Received 7 January 2013
Received in revised form
12 September 2013
Accepted 14 September 2013
Available online 23 September 2013

Keywords:
Capillary–gravity wave
Elliptic cylindrical container
Mathieu equation
Viscous frequency shift

ABSTRACT

The linear theory of unforced surface gravity–capillary waves in cylindrical containers with an elliptical cross-section is studied in detail. General solutions for the velocity potential and the free surface amplitude are given in terms of Mathieu functions. Our numerical results show the dependence of the natural frequencies on the fluid properties and the eccentricity e of the container cross-section. The well-known case of a circular tank for $e = 0$ is retrieved and remarkable crossings of the mode frequencies for certain values of e are found. The frequency shift and the wall damping ratio due to viscous dissipation in the Stokes boundary layers are evaluated numerically. The effect of the viscous dissipation in the bulk, the wall damping ratio, is estimated.

1. Introduction

An enormous amount of experiments and theoretical works have been devoted to study liquid surface waves in partially filled containers (see Refs. [1,2] and references therein). The theory of liquid surface waves dynamics is based on developing the fluid field equations, estimating the fluid free-surface motion, and the resulting hydrodynamic forces and moments. The modal analysis of a liquid free-surface motion in a partially filled container estimates the natural frequencies and the corresponding mode shapes. Knowledge of the natural frequencies is essential in the design process of liquid tanks and in implementing active control systems in space vehicles [1,2]. Explicit solutions are possible only for a few special cases such as upright cylindrical and rectangular containers. An example is the theoretical prediction of the natural frequencies when the contact line is pinned (pinned-end edge condition) [3,4]. This interesting problem with edge constraints, for the case of a circular cylinder, has yielded an excellent agreement between experimental and theoretical data. In particular, the agreement for damping rate calculations with experimental values is rather satisfactory in the case of pinned-end edges (when the dissipation in the bulk is taken into account) [5–8]. At least from the theoretical point of view, it would be beneficial to other geometry extend recent results achieved in circular geometry [9–12]. The natural geometry which can make such an extension is considering basins with an elliptical cross-section. However, even in the standard theory of standing gravity–capillary waves, where it is assumed that the contact line can move freely across the solid boundary (free-end edge condition), the results are poor in the elliptic geometry [13–15].

The aims of the present work are to present the effect of the elliptic geometry confinement on the natural frequency and to show that basins with elliptical cross sections can be used as a tool for modulation of the surface-wave damping. A brief outline of the paper is as follows. Section 2 presents the master equations and boundary conditions. The fluid is assumed to be an inviscid liquid and surface tension effects are considered. An exact solution of the problem for free-end edge, which involves Mathieu functions,

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http://dx.doi.org/10.1016/j.euromechflu.2013.09.003

is given in Section 3. Eigenvalues, nodal pattern and frequencies of the normal modes are given as functions on the eccentricity of the tank cross-section. Section 4 discusses the viscous dissipation ratio in the Stokes boundary layers on the lateral wall and the bottom. Finally, Section 5 estimates the interior damping ratio due to viscous dissipation in the bulk.

2. Governing equations

In the framework of an inviscid incompressible liquid and considering small oscillations at the surface, the three-dimensional velocity potential Φ of the transversal part of the propagating waves is given by [16]

$$\nabla^2 \Phi = 0, \tag{1}$$

where the potential Φ must fulfill the conditions

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} - \frac{\sigma}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial^2 \Phi}{\partial^2 x} + \frac{\partial^2 \Phi}{\partial^2 y}\right) = 0, \quad \text{at } z = 0 \tag{2}$$

$$\left.\frac{\partial \Phi}{\partial \mathbf{n}}\right|_{(x,y)\in\mathcal{L}} = 0, \tag{3}$$

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=-h} = 0, \tag{4}$$

with g being the acceleration of gravity, $\mathcal{L}$ the boundary of the container cross-section, $\mathbf{n}$ a unit vector perpendicular to contour line $\mathcal{L}$, h the fluid depth, σ the surface tension, and ρ_0 the fluid density.

In the case of a tank with an elliptical cross section the problem has axial symmetry; thus the stationary solution for $\Phi(\mathbf{r}, t)$ can be cast as

$$\Phi = \Psi(x, y)\cosh k(z + h)\exp(iw_0 t), \tag{5}$$

where w_0 is the natural frequency and k the wave number. From Eqs. (1)–(5) we obtain for $\Psi(x, y)$ the Helmholtz equation

$$\nabla^2 \Psi + k^2 \Psi = 0, \tag{6}$$

with the dispersion relation

$$w_0 = \sqrt{\left(g + \frac{\sigma}{\rho_0}k^2\right)k\tanh(kh)}. \tag{7}$$

Notice that the transversal amplitude of liquid surface at $z = 0$, denoted by $\zeta(\mathbf{r}, t)$, is related to the two-dimensional velocity potential Ψ by the equation [16]

$$\zeta = i\bar{\zeta}(x, y)\exp(iw_0 t), \quad \bar{\zeta} = \frac{w_0\cosh(kh)}{g + \sigma k^2/\rho_0}\Psi(x, y). \tag{8}$$

In that sense the nodal structure of free surface displacement is linked to the function $\Psi(x, y)$.

In the next section we will describe the solution of Eq. (6) by studying the main characteristics of the capillary–gravity waves in a basin with elliptical symmetry.

3. Inviscid solution

The Helmholtz equation can be written in elliptic coordinates as

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} + \frac{\varrho^2}{2}(\cosh 2\xi - \cos 2\eta)k^2\Psi = 0, \tag{9}$$

where $0 \le \xi, 0 \le \eta < 2\pi, \varrho > 0$, and elliptic and rectangular Cartesian coordinates are related by the equation [17]

$$x = \varrho\cosh\xi\cos\eta, \qquad y = \varrho\sinh\xi\sin\eta,$$

such that the curves $\xi = $ const are confocal ellipses and the curves $\eta = $ const are confocal hyperbolas, with focus at $(\pm\varrho, 0)$.

In our case we are looking for solutions of Eq. (9) in the interior of the domain $D = (\xi, \eta) : 0 \le \xi \le \xi_0, 0 \le \eta < 2\pi$, whose boundary $\mathcal{L}$ is an ellipse with semi-major (minor) axis A (B). Eq. (9) is factorized as $\Psi(\xi, \eta) = NF(\xi)G(\eta)$, where N is a normalization constant and the functions $F(\xi)$ and $G(\eta)$ satisfy

$$\left[\frac{d^2}{d\xi^2} - \alpha + \frac{\varrho^2 k^2}{2}\cosh 2\xi\right]F = 0, \tag{10}$$

$$\left[\frac{d^2}{d\eta^2} + \alpha - \frac{\varrho^2 k^2}{2}\cos 2\eta\right]G = 0, \tag{11}$$

with α the separation constant.

From physical considerations the function $G(\eta)$ must be periodic, i.e.,

$$G(\eta) = G(\eta + 2\pi). \tag{12}$$

Eq. (11) and the condition (12) are linked to Mathieu's equation, where α is a countably infinite set of characteristic values. These values can be denoted by $a_m(q)$ for the even solutions $G_e = ce_m(\eta, q)$ and by $b_m(q)$, for the odd ones $G_o = se_m(\eta, q)$, where $4q = \varrho^2 k^2$. The function $F(\xi)$ corresponds to the modified Mathieu's equation and it is related to $G(\eta)$ solution by the substitution $\eta = i\xi$ so that

$$F_e = ce_m(i\xi, q) = Ce_m(\xi, q), \quad (m = 0, 1, 2 \ldots), \tag{13}$$

$$F_o = -ise_m(i\xi, q) = Se_m(\xi, q), \quad (m = 1, 2 \ldots), \tag{14}$$

where Ce_m and Se_m are the modified Mathieu functions of first kind or radial solutions. Following the spatial symmetry of Eq. (9), the complete set of solutions can be chosen as even or odd eigenfunctions

$$\Psi_m^e = N_m^e ce_m(\eta, q)Ce_m(\xi, q), \tag{15}$$

$$\Psi_m^o = N_m^o se_m(\eta, q)Se_m(\xi, q). \tag{16}$$

3.1. Eigenvalues

In elliptic coordinates the condition $\partial\Psi/\partial\mathbf{n} = 0$ at $(x, y) \in \mathcal{L}$ reduces to

$$\left.\frac{dF}{d\xi}\right|_{\xi=\xi_0} = 0, \tag{17}$$

with $\xi_0 = \operatorname{arctanh}(B/A)$. Let $\varrho k_m = e\tilde{k}_m$, e being the eccentricity and $\tilde{k}_m = Ak_m$, then from Eq. (17) we obtain the set of even and odd eigenvalues for $\tilde{k}_{m,n}$, where $\tilde{k}^e_{m,n}$ and $\tilde{k}^o_{m,n}$ are solutions of

$$Ce'_m(\xi_0, (e\tilde{k}^e_{m,n}/2)^2) = 0, \qquad Se'_m(\xi_0, (e\tilde{k}^o_{m,n}/2)^2) = 0, \tag{18}$$

with $n = 1, 2, \ldots$. Recall that the eccentricity e of an ellipse is related to its semi-axes by the expression $e = \sqrt{1 - B^2/A^2}$ [17], so that one can write $\xi_0 = \operatorname{arctanh}(\sqrt{1 - e^2})$, and therefore, the characteristic values $\tilde{k}_{m,n}$ depends solely on the eccentricity e. Thus, if the ratio value between semi-axes B/A is modified, the pattern of the wave amplitude will be different carrying the information of the surface symmetry and thereby of elliptic geometry.

An important conclusion emanating from the Helmholtz equation in an elliptical domain is the separation of the eigenfunctions into two independent Hilbert subspaces, which we have denoted by O (odd) and E (even). The eigenvalue problem (18) states that for a given m the symmetry (O or E) remains valid for any eccentricity value. Thus, eigenvalues $k_{m,n}$ for even and odd solutions with the same m but different elliptical oscillation number n cannot cross