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The influence of surface tension upon trapped waves and hydraulic falls

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HIGHLIGHTS

- We model the free surface of an ideal fluid flowing past submerged obstacles.
- We examine the effect of surface tension and obstacle position, on the free surface.
- For hydraulic falls we show that solutions are not unique.
- We find gravity–capillary waves trapped between two obstructions.

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ABSTRACT

We consider steady two-dimensional free-surface flows past submerged obstructions on the bottom of a channel. The flow is assumed to be irrotational, and the fluid inviscid and incompressible. Both the effects of gravity and surface tension are considered. Critical flow solutions with subcritical flow upstream and supercritical flow downstream are sought using fully nonlinear boundary integral equation techniques based on the Cauchy integral formula. When a second submerged obstruction is included further upstream in the flow configuration in the absence of surface tension, solutions which have a train of waves trapped between the two obstacles before the critical flow have already been found (Dias and Vanden-Broeck 2004 [2]). We extend this work by including the effects of surface tension. Trapped wave solutions are found upstream for small values of the Bond number, for some values of the Froude number. Other types of trapped waves are found for stronger tension when the second obstruction is placed downstream of the hydraulic fall generated by the first obstacle.

1. Introduction

The problem of a free-surface flow past a disturbance in a channel, is a widely studied area in fluid mechanics. There are many naturally occurring physical situations that such a problem can model, as the disturbance can take one of many forms: a fully submerged obstruction at the bottom of the channel, e.g. the flow generated by a rock on a river bed, a submerged obstruction not touching the channel bottom such as a submarine moving under water, a surface piercing obstruction, e.g. the flow due to a ship moving through water, or a localised pressure distribution on the free surface (see Părău and Vanden-Broeck [1]), e.g. atmospheric disturbances caused by high winds. Both steady and unsteady solutions to this problem are known to exist, but we consider just the steady case. We concentrate our attention on flow past submerged obstructions.

Solutions for the pure gravity wave problem have been obtained using fully nonlinear methods. In the case of a single submerged obstacle on the bottom of the channel, four different types of basic solutions are known to exist (Dias and Vanden-Broeck [2]). These solutions depend on the Froude number F, defined by

$$F = \frac{U}{\sqrt{gH}} \tag{1}$$

where U is the downstream velocity of the fluid, g is the gravitational acceleration, and H is the downstream depth of the fluid. When $F > 1$ the flow is said to be supercritical, and when $F < 1$, subcritical. We also define the upstream Froude number F_{up}, by

$$F_{up} = \frac{V}{\sqrt{gh}} \tag{2}$$

where V is the upstream velocity of the fluid, and h is the upstream fluid depth. The first type of basic solution is then classified as having uniform supercritical flow both up and downstream of the obstacle, with a forced solitary wave over the obstruction. If the

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obstacle itself has vertical symmetry about its centre, the free surface will then also be symmetric about the obstacle. The second type is subcritical and consists of a uniform flow upstream, with a train of waves downstream of the obstacle. For both these solutions, the mean depth of the fluid in the linearised theory, is the same both up and downstream. Forbes and Schwartz [3] used a boundary integral technique to obtain type one and two solutions for flow over a semi-circular obstruction. Vanden-Broeck [4] then found that flows of type one are not unique. There exist two solutions for particular values of the Froude number; a perturbation from a pure solitary wave, and a perturbation from the uniform stream.

The last two types of flow are critical. The first is a hydraulic fall, which consists of a subcritical uniform flow upstream, with a gradual increase in the Froude number over the obstacle, resulting in supercritical uniform flow downstream. The change in the Froude number from subcritical to supercritical means that the depth of the fluid decreases over the obstacle as we travel downstream. Forbes [5] computed such a flow configuration over a semi-circular obstacle using boundary integral equation techniques, and Dias and Vanden-Broeck [6] used a series truncation method to obtain solutions over a triangular obstacle. It was found that as the size of the obstacle increased, the downstream Froude number increased, while the upstream Froude number tended to zero.

Generalised hydraulic falls were first computed numerically by Dias and Vanden-Broeck [2]. These are like hydraulic falls but with a train of waves upstream of the obstacle. However, this final type of solution is unphysical when considered as the free-surface flow over a single obstruction, as it does not satisfy the radiation condition, requiring that there is no energy coming from infinity (and thus, no waves upstream of the obstacle). Hydraulic falls have only been observed with subcritical flow upstream. The direction of the flow for the generalised hydraulic fall can therefore not simply be changed, so that the flow upstream is uniform and supercritical, in order to satisfy the radiation condition. However, Dias and Vanden-Broeck [7] have shown that this flow can become physically relevant, when considered as the localised flow over an obstacle, in a configuration involving at least one other obstacle further upstream. The free surface they obtained is a hybrid solution of the second type of basic flow over the obstacle upstream and the generalised hydraulic fall over the second obstacle. Such solutions over multiple obstructions have also been observed experimentally, for example, by Pratt [8] who found that the length and amplitude of the trapped waves between the two obstacles remains unchanged when the distance between the obstacles is increased. Only the number of waves between the obstacles changes. Pratt also found that the solution is characteristic for different shaped obstacles. The shape of the obstacle affects only the amplitude and wavelength of the trapped waves.

Binder, Dias and Vanden-Broeck [9] used both weakly nonlinear and fully nonlinear techniques to consider the possible configurations for flow past two submerged obstructions. They found solutions which have a train of waves upstream of the first obstacle, but uniform subcritical flow between the obstructions, and supercritical uniform flow downstream of the second obstruction. Although not physically relevant due to violating the radiation condition, they noted that such solutions could be made physically relevant, by introducing further obstacles upstream. Belward [10] looked at the solutions obtained when a second obstacle occurs downstream of the hydraulic fall. A forced solitary wave exists over the downstream obstacle, and Belward found that the speed of the flow is almost entirely determined by the hydraulic fall.

There are fewer studies which include the effects of surface tension. To characterise gravity–capillary waves, we introduce the Bond number

$$\tau = \frac{T}{\rho g H^2}. \tag{3}$$

Here T is the value of the tension on the free surface and ρ the density of the fluid. Forbes [11] obtained fully numerical solutions for flow over a semi-circular obstacle when the effects of both gravity and surface tension are included. As well as obtaining solutions of type one, Forbes obtained solutions with a train of capillary waves upstream, and gravity waves downstream. Grandison and Vanden-Broeck [12] also studied this type of solution, and removed the inaccuracies in the solutions caused by truncation of the flow both up and downstream.

Maleewong, Asavanant and Grimshaw [13,14] used both fully and weakly nonlinear methods to examine the forced solitary waves produced by a positively and a negatively orientated single applied pressure distribution. However, they restricted their study to the first type of basic solution, in the presence of a single disturbance. Page, Părău and Grandison [15] then studied gravity–capillary forced solitary waves generated by two localised pressure distributions, and Guayjarernpanishk and Asavanant [16] included the effects of surface tension in their study, and found basic flows of types one, two and three. In this paper, we include the effects of surface tension, and consider types three and four of the basic solutions; hydraulic falls and generalised hydraulic falls, as well as the effect of a second disturbance in the channel.

In the next section, we mathematically formulate the problem. The results are presented in Section 3, and in the final section, Section 4, we conclude with a summary of our results.

2. Formulation

We consider the free surface of an inviscid, incompressible fluid flowing along a channel. The flow is assumed to be steady and irrotational, and is subject to gravitational acceleration in the negative y^*-direction. On the channel bottom there exists one or multiple submerged obstructions. We introduce Cartesian coordinates x^*, y^* and align the x^*-axis so that it is parallel to the bottom of the channel, with the y^*-axis directed vertically upwards, through one of the obstructions.

We let $y^*(x^*) = H + \eta^*(x^*)$ define the free-surface elevation, and take $y^* = B^*(x^*)$ to be the function describing the bottom of the channel. The flow is assumed to be uniform in the far field, as $x^* \to \pm\infty$, with constant depth H, and constant velocity U downstream, and constant depth h and constant velocity V upstream. The downstream Froude and Bond numbers are then given by Eqs. (1) and (3) respectively. The upstream Froude number is given by Eq. (2), and we introduce the upstream Bond number

$$\tau_{up} = \frac{T}{\rho g h^2}. \tag{4}$$

To continue, we non-dimensionalise by taking U as unit velocity, and H as unit height. Non-starred variables are thus now understood to be dimensionless. We define the dimensionless upstream velocity by γ, and so using conservation of mass, the dimensionless upstream depth is $\frac{h}{H} = \frac{1}{\gamma}$. The dimensionless fluid system is shown in Fig. 1.

The problem is formulated as a system of nonlinear equations in terms of the velocity potential $\phi(x, y)$, which, must satisfy Laplace's equation in the flow domain

$$\phi_{xx} + \phi_{yy} = 0 \tag{5}$$

and the corresponding boundary conditions. The kinematic conditions on the free surface $y = 1 + \eta(x)$ and the channel bottom $y = B(x)$ are

$$\phi_y = \phi_x \eta_x \quad \text{and} \quad \phi_y = \phi_x B_x \tag{6}$$

respectively. Applying Bernoulli's equation on the free surface, we obtain the dynamic boundary condition

$$\phi_x^2 + \phi_y^2 + \frac{2}{F^2} y = \frac{2}{F^2} \tau\kappa + 1 + \frac{2}{F^2} \tag{7}$$