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Spatially periodic temperature modulation of a compressible fluid

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ABSTRACT

A spatially periodic temperature modulation is gradually applied at the lower boundary of a layer of compressible fluid. The temperature from the lower wall diffuses into the layer and induces various convection patterns. As the amplitude of the temperature modulation is increased, non-linear effects, including those due to the inclusion of compressibility, become more prominent. An accurate numerical scheme is developed to capture the full time-dependent behaviour here. Spectral methods will be used throughout this work to provide accurate representations of the various solution components and allow for the efficient implementation of a variety of boundary conditions.

Three different types of modulation are considered, namely a pure cosine as well as rounded triangle and rounded square profiles, where the latter two of these have applications in various physical situations. Interest lies in how the nature of the convection and temperature diffusion change as the amplitude of these modulations is increased. Both no-slip and slip conditions will be implemented on the upper and lower boundaries of the layer and the differences between the two will be considered for selected cases.

1. Introduction

The behaviour of a compressible gas subjected to non-homogeneous heating from below will be studied. In some sense this could be considered a variation on the Rayleigh–Bénard problem, in which a fluid is uniformly heated from below and cooled from above. The points of interest here, namely the use of a compressible gas and the non-homogeneous heating, appear not to have been studied extensively in combination. When considered separately these two effects are applicable to various meteorological and astrophysical contexts, and by combining them here new aspects of these problems are revealed. A linearized version of this problem, where the amplitude of the non-homogeneous part of heating is assumed to be small, will be considered elsewhere by Chen and Forbes [1]. That investigation is particularly concerned with determining the effect of parameters such as layer height and viscosity on the flow.

The extension of the classic Rayleigh–Bénard problem into compressible flow is a logical progression, particularly given the application of compressible flows to fields such as astrophysics and meteorology. Some work by Spiegel [2], and continued by Gough et al. [3], modelled the processes of stellar convection as a compressible gas heated from below. The same concerns that are the focus of much of the fundamental work on the Boussinesq version of the flow, those of determining the critical point at which convection occurs as well as the nature of this convection, are also at the fore in the analysis of the compressible problem. In particular, the linear analyses of Spiegel [2], Gough et al. [3] and Gauthier et al. [4] are concerned with investigating the onset of convection in the two-dimensional case, and these studies make use of a variety of numerical and asymptotic techniques. A linear analysis of the equivalent problem in three dimensions by Bormann [5] obtained critical values for a variety of compressible gases.

A number of studies have investigated the non-linear behaviour of these problems, typically with an emphasis on effects that are present in the compressible flow, but not in the Boussinesq version. A number of these effects are detailed by Furukawa and Onuki [6], whose focus is on the process of heat transport, and reveal the existence of various transient and steady solutions as well as noting some unique behaviours for highly compressible fluids. Similarly, Manela and Frankel [7] develop solutions that are characterized by convective motion in a small region near the walls. A comparison between slip and no-slip wall conditions has been made by Gauthier [8] and this comparison will be used in the work here as well. That paper made extensive use of spectral methods and the formulation used there is similar to that adopted in the present work.

Another relevant extension of the Rayleigh–Bénard problem involves the spatial modulation of the conduction solution. A stability analysis for this problem has been conducted by Kelly and Pal [9] and a key result of that study was in establishing the connection between temperature modulation and spatially undulating walls. Further work on the exact nature of the associated steady convection for this type of flow has been performed by Riahi [10] who also focussed on the effect of modulating temperature at

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both the upper and lower walls where these modulations were allowed to be of different amplitudes. The combination of undulating boundaries and a temperature modulation has been studied by Schmitz and Zimmermann [11] who, among other results, revealed the presence of a Hopf bifurcation as the amplitude of the temperature modulation was increased. A more recent study is that of Freund et al. [12] who examined the various types of instability and pattern forming behaviour that may occur when a periodic temperature modulation is applied at the lower wall. This is very similar to the problem to be considered in this paper, except that here the temperature modulation will be examined in isolation from the uniform temperature gradient used in these works.

The key driver of the flow in this paper will be a non-uniform temperature applied at the lower wall. Previous authors have typically made use of either a Boussinesq or an anelastic approximation in treating this type of flow. There are a number of possible choices for the precise form of the inhomogeneity in temperature at the lower wall, and several will be considered in the course of the work here. A straightforward choice is a sinusoidal variation, as was adopted by Somerville [13], whose work used spectral methods and revealed the presence of asymmetrical solutions and the importance of non-linear effects. Rossby [14] considered a linearly varying temperature on the lower wall. The focus of that work was on the resulting convective overturning behaviour in the presence of density stratification, with a particular emphasis on the application of such behaviour to ocean mixing.

Spectral methods will be employed in a number of contexts in this paper and these will provide accurate and efficient techniques to compute various solution components. Some previous work on the compressible version of the Rayleigh–Bénard problem by Gauthier [15] made use of such methods, with an emphasis on choosing collocation points and applying implicit techniques. Earlier authors such as Lorenz [16] and Somerville [13] similarly considered heated flows using low-order spectral methods, and these were useful in elucidating key behaviours, qualitatively describing many aspects of the flow and even making reasonable approximations to some non-linear effects.

The work presented here models the non-uniform heating from below of a compressible fluid using a variety of techniques. The formulation of the problem is presented in Section 2, including a discussion of boundary conditions and the related conduction solution. The numerical solution technique for this problem is outlined in Section 3 and these solutions are compared to the linearized solutions obtained separately in Chen and Forbes [1], where the amplitude of the temperature modulation is assumed to be small. In particular, the effect of increasing the amplitude of this non-uniform part of temperature for a variety of temperature modulations will be considered here. Additionally, a comparison of slip and no-slip boundary conditions will be made for selected cases.

2. Formulation & non-dimensionalization

The flow of a compressible perfect gas, confined between two plates of infinite lateral extent, under non-homogeneous heating from below is considered. It is assumed that the flow is two dimensional and that there is no net heat flux in or out of the system. The exact form of the boundary condition on the lower wall will be specified in quite a general way so that a variety of profiles for the non-homogeneous part of lower wall temperature may be studied.

A schematic diagram of the flow is shown in Fig. 1. The fluid region of interest is contained within a box of height h, and there are open vertical boundaries on the left and right; later, the flow will be assumed to be periodic in the x-direction. The upper wall is held at a fixed temperature, whilst an inhomogeneous temperature

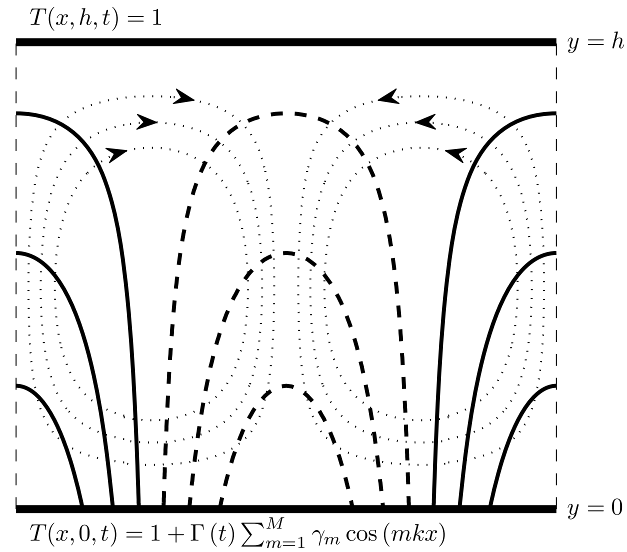

Fig. 1. Schematic diagram of the flow. The temperature field is displayed as solid and dashed lines, where the solid lines represent temperature greater than the ambient temperature and the dashed lines are less than the ambient temperature. The light dotted lines are streamlines of velocity and the arrows indicate the direction of rotation for each of the convection cells. All quantities are dimensionless.

profile is applied to the lower wall. Details of the exact form of the inhomogeneity will be given presently. The fluid has an ambient, undisturbed temperature of T_1 and it is assumed that the lower wall is at some reference atmospheric pressure p_{atm}. There is a body force due to gravity g, acting downward, which plays an important role in buoyancy effects.

The flow is modelled with the full Navier–Stokes equations for a compressible gas, as well as an energy equation. Here it is assumed that the viscosity and thermal diffusivity are constant. Following Tannehill, Anderson, and Pletcher [17], the equations for conservation of mass, x- and y-momentum and energy are derived. The dimensional versions of these equations are

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{q})=0 \tag{2.1}$$

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t}=-\frac{\partial p}{\partial x}+\mu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial x\partial y}\right) \tag{2.2}$$

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t}=-\rho g-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{4}{3}\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial y}\right) \tag{2.3}$$

$$\rho\frac{\mathrm{D}E}{\mathrm{D}t}=\kappa\nabla^2 T-p\left(\nabla\cdot\mathbf{q}\right)+\mu\Phi \tag{2.4}$$

where we have defined a velocity vector $\mathbf{q}=u\mathbf{i}+v\mathbf{j}$, and $\mathrm{D}/\mathrm{D}t=\partial/\partial t+\mathbf{q}\cdot\nabla$ is the usual material derivative. The constant μ is the dynamic viscosity, κ is the thermal conductivity and g is acceleration due to gravity. A viscous dissipation term Φ is included in the energy equation (2.4) and is defined as

$$\Phi=\frac{4}{3}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)+\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\left(\frac{\partial u}{\partial y}\right)^2 \tag{2.5}$$

although in practice this term is small compared to the rest of Eq. (2.4). The fluid under consideration is assumed to be a monatomic perfect gas and it follows that energy and pressure are defined by the thermodynamic relations

$$E=c_V T \tag{2.6}$$

$$p=\rho RT \tag{2.7}$$