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Stochastic evolution equations with localized nonlinear shoaling coefficients

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ABSTRACT

Nonlinear interactions between sea waves and the bottom are a main mechanism for energy transfer between the different wave frequencies in the near-shore region. Nevertheless, it is difficult to account for this phenomenon in stochastic wave models due to its mathematical complexity, which consists of computing either the bi-spectral evolution or non-local shoaling coefficients. Recent advances allowed the localization of the nonlinear shoaling coefficients, setting a simpler way to apply this mechanism in these models for one-dimensional interactions. This was done by taking into account only mean energy transfers between the modes while neglecting oscillatory transfers. The present work aims to improve these localized coefficients in order to make them more consistent with the dominating resonance mechanism—the class III Bragg resonance. The approximated stochastic models are tested with respect to a deterministic nonlinear mild-slope equation model, where a significant advantage of the improved coefficients is observed.

1. Introduction

Nonlinear energy transfer is a dominant process that affects the evolution of wave spectra both in deep water and in the shoaling region. The nonlinear interactions in deep water consist of wave quartet interactions at leading order. These wave quartets, which act at cubic nonlinearity in wave steepness, satisfy resonant conditions of the wave frequencies and wave numbers. This type of evolution is rather a weak one that requires large spatial distances (time) of thousands of wavelengths (wave periods) in order to have a considerable effect. In intermediate to shallow water, the nonlinear interactions act much faster with significant energy transfers between triads of waves. This is possible due to the influence of the bottom that enables us to satisfy the resonant conditions already in quadratic nonlinearity. Furthermore, when waves shoal their steepness increase, and as nonlinear interactions are proportional to the wave steepness, the nonlinear energy transfer becomes even larger in this region.

Various wave models address the problem of nonlinear interactions in the near shore environment. Boussinesq-type equations reduce one spacial dimension assuming the depth is small compared to the wavelength. These equations can compute the nonlinear time-domain problem with great accuracy (see, e.g. [1]), but result in a very high computer effort. Other methods assume a set of slowly evolving harmonic wave components with a vertical profile that fits the linear motion over a flat bottom (mild-slope-type assumptions). This approach results in a set of evolution equations for each harmonic that are coupled with quadratic nonlinear terms. These equations can be hyperbolic (e.g. [2]), elliptic and parabolic (e.g. [3–5]).

The advantage of using a stochastic approach is the significant reduction in calculation effort, as the Nyquist limitation no longer restricts the numerical solution. Several works on stochastic wave models that account for nonlinear interactions were presented. Agnon and Sheremet [6,7], Kofoed-Hansen and Rasmussen [8], Eldeberky and Madsen [9] presented stochastic evolution equations based on hyperbolic models taking into account one-dimensional interactions. Herbers and Burton [10] derived stochastic evolution equations starting from a Boussinesq-type model while presenting as well two-dimensional calculations for the quasi-one-dimensional problem (no bottom changes in the lateral direction). Janssen et al. [4] derived a stochastic model, using a Fourier transform in the later direction. Their model includes diffraction effects, while accounting for two-dimensional quadratic nonlinear interactions that allow mild changes in the lateral direction.

The common and most widely used forecasting models are based on a stochastic hyperbolic wave action equation. In these models, simplified one-dimensional parametric source functions are used to describe the triad interactions (see [11,12]). In these source functions there is only energy transfer to higher harmonics of each spectral component (self-interactions) without accounting for other transfers of energy of different triad combinations and energy that is transferred to lower harmonics. This approach

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enables an easy inclusion of simplified nonlinear energy transfers, but may lead to a physically inappropriate evolution of the spectrum (see [13,12]). In addition, the derivation of parametric source functions consists of a local flat bottom approximation. As oscillating bottom components, which enable satisfaction of wave number resonance conditions, come as well from the bottom profile's derivatives, it inherently fails to accurately model the dominating energy transfer—the class III Bragg resonance.

Here lies a wide gap. On one hand there are stochastic models such as [4] that take into account the directional spreading of the triad interactions, but on the other hand this physics is not applied to the wave forecasting models even for the one-dimensional case. The problem in including the two-dimensional quadratic interaction model of [4] is that it is based on a Fourier transform in the lateral direction that poses a problem in applying it to the hyperbolic formulation of the wave action equation models. Furthermore, for these forecasting models even the calculation of the one-dimensional quadratic interactions is costly for its inclusion, as these models are run in real time for very large spacial and temporal domains. Hence, a lighter quadratic nonlinear model is required in order to present an alternative approach that still grasps the essence of this important phenomenon.

Among the aforementioned stochastic works a main advancement in reducing the bi-spectral calculation costs was made by Agnon and Sheremet [6]. They presented an analytical definition of the bi-spectra that allows its substitution into the evolution equations without the need for its direct numerical calculation. Still, due to this operation the resulting interaction coefficients became non-local, and therefore, difficult to apply to forecasting models.

In a later work, Agnon and Sheremet [7] improved the accuracy of the nonlinear triad interactions. In addition, they localized the non-local coefficients by assuming the bottom to be a sum of oscillating components. More recent progress was made by Stiassnie and Drimer [14], who managed to localize the non-local shoaling coefficients of [6] by neglecting harmonic back and forth energy transfers between the modes and accounting only for the mean energy transfer. This progress paves the way for applications of this approach for one-dimensional interactions also in two-dimensional wave action equation-type forecasting models, as it significantly lowers the computational effort, while still incorporating the mean energy transfer between all wave triad combinations.

The present work aims to apply the method of Stiassnie and Drimer [14] to the more accurate one-dimensional non-local shoaling coefficients of Agnon and Sheremet [7]. This simplistic approach is not supposed to compete with more accurate models such as the ones of Herbers and Burton [10], Agnon and Sheremet [7] and Janssen et al. [4] but rather improve another line of work—the simpler localized nonlinear interaction terms appropriate for wave forecasting models given by Elderbeky and Battjes [11], Becq-Girard et al. [12] and Stiassnie and Drimer [14].

The paper is constructed as follows. In Section 2, an overview is given on resonant interactions in the near-shore region. The stochastic model of Agnon and Sheremet [7] is presented in Section 3 together with the non-local nonlinear shoaling coefficients. In Section 4 the non-local shoaling coefficients of [7] are inspected. Then, new local shoaling coefficients are derived and compared together with the coefficients of [14] to the non-local coefficients of Stiassnie and Drimer [7]. Numerical calculations are presented in Section 5, and the work is summarized in Section 6.

2. Resonant interactions

In order to better understand the nonlinear interactions in the shoaling region, it is helpful to observe the problem in the frequency and wavenumber domains with respect to resonant interactions. These resonant interactions (as well as near resonant ones) represent the majority of energy transfer within the wave spectrum. For a wave field in deep water, interactions among different wave components become resonant at order m (in wave steepness), if the wavenumbers k_j and the corresponding frequencies ω_j satisfy resonance conditions. This requires the sum of wavenumbers and frequencies to satisfy the following relations

$$\omega_1 \pm \omega_2 \pm \cdots \pm \omega_{m+1} = 0, \qquad \mathbf{k}_1 \pm \mathbf{k}_2 \pm \cdots \pm \mathbf{k}_{m+1} = 0, \quad m \geq 1. \tag{1}$$

As the wave number and the frequency of each wave are related through the dispersion relation, the satisfaction of Eq. (1) in deep water can not occur at $m = 2$ (i.e. between wave triads). Therefore, the leading order interaction is of a quadruplet of waves at $m = 3$, which is supplemented by weaker interactions at $m = 4, 5, \ldots$. In shallow to intermediate waters, a bottom-induced free-surface interference, which does not abide by the dispersion relation, can allow the satisfaction of this resonance relation (1) even at order $m = 1$. These resonant interactions, which consists of bottom components in addition to surface wave ones, relate to the so-called Bragg resonance.

The linear class I and class II Bragg resonances occur at order $m = 1$ with one bottom component and with two bottom components respectively. The nonlinear class III Bragg resonance occurs at order $m = 2$ with one bottom component. The class I and class II Bragg resonances are the wavenumber representation of the main linear reflection and refraction effects, whereas the class III Bragg resonance is the main wavenumber representation of the nonlinear triad interactions in shallow to intermediate depths. Eq. (1) can be used to describe higher orders of linear and nonlinear interactions with more bottom components, but these interactions usually have a lesser effect. Different terms in wave equations can be ordered using this classification. For simplification purposes, these equations can be truncated in a consistent way above a chosen Bragg class resonance order.

For $m = 2$ with one bottom component, Eq. (1) takes the form:

$$\omega_1 \pm \omega_2 \pm \omega_3 = \gamma, \qquad \mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{K}_n = \delta. \tag{2}$$

Here, $\mathbf{K}_n$ is a bottom component, and small detuning parameters, δ and γ, have been added in order to represent the near resonant interactions. Eq. (2) describes the class III Bragg resonance conditions.

3. Stochastic models

In this section the development of [6,7] is presented. For the one-dimensional case Agnon and Sheremet obtained the following equation:

$$\frac{d\left\langle \left|B_f\right|^2 \right\rangle}{dx} = 2\sum_{f_1}\sum_{f_2} \tilde{W}_{(0,1,2)} \Im m \times \left[\left\langle B_f^* B_{f_1} B_{f_2} e^{-i\int_{-\infty}^{x} \Delta_{0:1,2} dx'}\right\rangle\right] \delta_{f,f_1+f_2} - 2\sum_{f_1}\sum_{f_2} \tilde{W}_{(0,-1,2)} \Im m \times \left[\left\langle B_f^* B_{f_1}^* B_{f_2} e^{-i\int_{-\infty}^{x} \Delta_{2:0,1} dx'}\right\rangle\right] \delta_{f_2,f+f_1}. \tag{3}$$

Here,

$$B_f = Cg_f^{1/2} A_f \tag{4}$$

with Cg_f as the modal group velocity from linear theory and A_f as the modal amplitude. The notation $\langle \cdots \rangle$ represents an ensemble