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Film flow for power-law fluids: Modeling and linear stability

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ABSTRACT

The paper deals with modeling of a power-law fluid film flowing down an inclined plane for small to moderate Reynolds numbers. A model, accurate up to second order [first order] for dilatant [pseudoplastic] fluids is proposed to describe the nonlinear behavior of the flow. The modeling procedure consists of a combination of the lubrication theory and the weighted residual approach using an appropriate projection basis. A suitable choice of weighting functions allows a significant reduction of the dimension of the problem. The resulting model is naturally unique, i.e., independent of the particular form of the trial functions. Reduced models are proposed for the evolution of the local film thickness and flow rate; their linear spectra are compared to that obtained from the full Orr–Sommerfeld numerical solution. To obtain the latter, a new formulation of the eigenvalue problem is proposed to overcome the classical divergence of the apparent viscosity at the free surface. The full model and its reduced forms all have the advantage of the Benney like model close to criticality. Far from the instability threshold the full model continues to follow the Orr–Sommerfeld solution up to sufficiently large Reynolds numbers and gives better predictions than the depth averaging model. An incomplete regularization procedure is performed to cure the rapid divergence of the reduced two-equation model. Due to its relative simplicity the latter might be preferred in practice to the full model, at least at the initial stage of the nonlinear regime. It is also shown that the convective nature of the instability is not affected by the variation of the power law index.

1. Introduction

Falling films on inclined planes are driven by a gravitational pressure gradient and become unstable when inertia overcomes hydrostatic pressure effects. The disturbance originates at the free surface where vorticity is produced by the base flow shear stress. Because of its advection by the base flow, the produced vorticity becomes out of phase with the disturbed interface so as to cause the amplification of the interface disturbance. The base flow then undergoes a Hopf bifurcation leading to periodic two dimensional nonlinear waves which evolve into a variety of patterns depending on the flow conditions. It is established both from numerical simulations and experiments that solitary wave structures play a central role in the long time behavior of the flow. As shown by the computations of Malamataris et al. [1], the velocity profiles beneath solitary waves are strongly deformed in comparison with the parabolic base flow velocity and the dynamics quite delicate. It is therefore useful for a fundamental understanding of the flow to develop reduced systems that can be exploited both analytically and numerically. Since the instability manifests itself in surface waves whose wavelength is, except for very small inclination, much larger than the film thickness, long waves of the Benney type [2] allow description of the flow development near criticality. The flow variables are then all enslaved to the local interface shape. Even though the Benney equation (BE) contains different physical mechanisms and is potentially capable of describing the near critical nonlinear behavior, it loses its physical relevance when the convective effects become significant, because of the production of shorter waves. The solutions of the BE then depart from those of the full Navier–Stokes equations and, at some distance beyond the stability threshold, they exhibit nonphysical finite time catastrophic behavior [3]. To overcome some of the drawbacks associated to the BE, several improvements were recently proposed. The regularization procedure developed by Ooshida [4] allows to avoid the occurrence of time blow up but fails to serve as an accurate model at moderate Reynolds numbers as its solitary wave solution exhibits unrealistically small amplitudes and velocities. Another single evolution equation including the second order dissipative effects via a suitable scaling was proposed by Panga and Balakotaiah [5]. Ruyer-Quil and Manneville [6] have shown that the Panga and Balakotaiah model can be modified such that its inertial terms correspond to Ooshida's equation. The failure of the long wave models to correctly describe nonlinear

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behaviors far from criticality is partly due to their incapacity to capture all of the inertia effects. The way to improve the modeling would be, according to Ruyer-Quil and Manneville [6] to incorporate the flow rate which becomes a genuine variable just after the wave formation. Such a model was first introduced by Shkadov [7] by using an integral boundary layer (IBL) approach. This theory combines the long wave assumption with the depth averaging method of Karman Polhansen type. In spite of its success to describe nonlinear regimes for moderate Reynolds numbers, the IBL approach does not accurately predict the flow behavior close to the stability threshold as the BE does. This defect is, as we will see later on, due to the fact that the IBL is coherent only up to zeroth order in the long wave parameter near criticality. A better account of the first order convective terms near criticality is therefore required to remove this drawback. The remedy was found by Ruyer-Quil and Manneville [8,9] by using a weighted residual integral boundary layer (WRIBL) approach. Their model, developed for both first and second order approximations, corrected the inability of the Shkadov approach to match the linear stability threshold and was found to yield bounded solutions for a larger range of Reynolds numbers than in the case of the BE. It has been demonstrated [8] that both first and second order models compared well with the experiments conducted with alcohol by Kapitza and Kapitza [10] on a vertical plane. However, only the second order model compared well with the experimental observation of Liu et al. [11] in water glycerin solution on an inclined plane. The WRIBL approach was extended to nonisothermal flows by Ruyer-Quil and co-workers [12,13] and to two layer flows by Amaouche et al. [14].

The objective of the present study is the derivation of nonlinear evolution equations for falling power law fluid similar to those obtained by Ruyer-Quil and co-workers [8,9,12,13] for Newtonian fluid. Indeed, it is important to understand how non-Newtonian effects affect the dynamics of falling film flows since they are present in a wide range of physical and technological applications. For instance, these specific effects are encountered in situations such as plastic manufacturing, coating processes, biological fluid motions, geological flows. Our interest is specifically with fluids whose rheological behavior can be described by a power law model which is a relatively simple constitutive equation. In that area, there are not so many investigations as in Newtonian case. Lin and Hwang [15] used the method of multiple scales to solve a nonlinear equation of Benney type. Their results indicate that subcritical instability and explosive solution occur at small power law index, supercritical and unconditional stable region exist solely when that index is greater than some critical value. A long wave of Benney type equation were also used and numerically integrated in periodic domain by Miladinova et al. [16] who found that the shape and amplitude of traveling waves are strongly dependent on the non-Newtonian properties of the fluid. The boundary layer integral method were applied by Dandapat and Mukhopadhyay [17] to derive an evolution equation for the so-called kinematic and inertial waves. They found, among other results, that the power law exponent plays a prominent role in controlling the surface tension effects. In a recent paper Sisoev et al. [18] present a bifurcation analysis of steady traveling waves by using a similar equation to that derived in [17]. Similar limitations to those described above were also encountered when using lubrication theory as well as Shkadov's procedure for non-Newtonian liquid film flows. To cure these limitations, we extend the idea developed in [12] for Newtonian fluids. The paper is organized as follows. Section 2 is devoted to the formulation of the governing equations. In Section 3, the problem is reformulated in terms of dimensionless boundary layer equations where third and higher order terms are neglected. The derivation of both first and second order system of evolution equations is performed in Section 4. More tractable models of reduced dimensionality are obtained in Section 5 and their linear stability is discussed in Section 6. Concluding remarks are presented in Section 7.

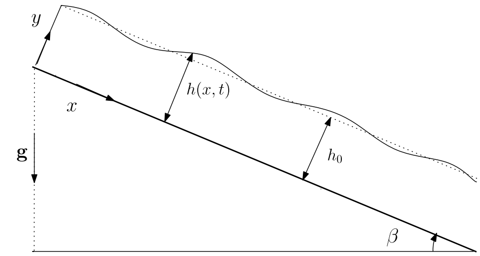

Fig. 1. Schematic of the physical problem.

2. Governing equations

The physical model of the problem is depicted in Fig. 1. A power law liquid of constant density ρ, consistency K and index n, flows under gravity along an infinitely long flat plate which is inclined at an angle β to the horizontal. A coordinate system (x, y) is adopted with x as the downstream coordinate and y being measured normal to the plate. The surface tension coefficient between the liquid and the surrounding passive medium (with pressure $p_a = 0$) is σ and the acceleration due to gravity is $\mathbf{g}$. Denoting by $\mathbf{v} = (u, v)$, p and $\boldsymbol{\tau}$ the velocity field, the pressure and the stress tensor, conservation of mass and momentum then read:

$$\operatorname{div}\mathbf{v} = 0, \tag{1}$$

$$\rho(\partial_t + \mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\mathbf{grad}p + \operatorname{div}\boldsymbol{\tau} + \rho\,\mathbf{g}, \tag{2}$$

where $\boldsymbol{\tau} = 2K\eta\mathbf{d}$ with $\eta = \dot{\gamma}^{n-1}$, $\dot{\gamma}$ denoting the second invariant $\sqrt{2d_{ij}d_{ij}}$ of the strain rate tensor $\mathbf{d}$. The index n indicates the degree of the non-Newtonian behavior and the greater is the departure from unity the more pronounced are the non-Newtonian effects, $n < 1$ corresponds to shear thinning (pseudoplastic) behavior while $n > 1$ represents shear thickening (dilatant) behavior. The above equations are subject to the boundary conditions

$$\text{on } y = 0 \qquad u = v = 0, \tag{3}$$

$$\text{on } y = h(x,t) \qquad h_t + u\,h_x - v = 0, \tag{4}$$

$$(-p + \sigma\operatorname{div}\mathbf{n})\mathbf{n} + \boldsymbol{\tau}\cdot\mathbf{n} = 0. \tag{5}$$

Expressing no slip on the rigid plate, the impermeability of the free surface $y = h(x, t)$ and equilibrium of all forces acting on it respectively, $\mathbf{n}$ being the unit vector normal to the free surface. The flat film solution for $h(x, t) = h_0$ has the form [16]

$$U_b = \frac{n}{n+1}\left(\frac{\rho\,g\sin\beta}{K}\right)^{1/n}\left[h_0^{1+1/n} - (h_0 - y)^{1+1/n}\right], \qquad V_b = 0. \tag{6}$$

This is analogous to the Nusselt solution for Newtonian fluid flow.

3. Dimensionless boundary layer equations

At low to moderate Reynolds numbers, the dominant instability of the flat film state is known to involve interfacial distortion and to have wavelengths much longer than the film thickness, except for slightly inclined planes (β less than about 1° for water) or for fluids with low surface tension. We will not consider these extreme conditions for which the instability changes to become a shortwave instability. So, to remove dimensions and express (1)–(5) in their dimensionless form we will assume two length scales, a typical wavelength l_0 and h_0 as the characteristic measures of distances downslope and transverse to the film respectively. The streamwise velocity u and the transverse velocity v are scaled with the depthwise averaged velocity $u_m = \frac{n}{2n+1}\left(\frac{\rho g \sin\beta}{K}\right)^{1/n} h_0^{1+1/n}$ and ϵu_m respectively, with $\epsilon = h_0/l_0$ being a stretch parameter