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Passive scalar advection in the vicinity of two point vortices in a deformation flow

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ABSTRACT

The dynamics of passive fluid particles in the vicinity of two point vortices with arbitrary intensities, embedded in a steady external deformation flow, is studied. The motion of passive fluid particles is described by a nonintegrable 1.5 degrees of freedom dynamical system. Though the external flow is stationary, the additional half degree of freedom appears because the vortices' motion about their stationary positions is periodic. Then, this periodic motion plays the role of a periodic perturbation for the system describing the passive particle dynamics. Therefore, chaotic advection of passive fluid particles in the vicinity of these two vortices can occur. If the vortices, however, are situated at their stationary positions, they become motionless, and the dynamical system describing the passive particles' dynamics is also stationary. In the case of motionless vortices, a classification of the phase portraits of the passive particle motion is conducted by analyzing the number of critical points. When the vortices do not lie at their stationary position, the system becomes nonstationary. In this case, the existence of impenetrable transport barriers for chaotic advection is shown. These barriers are destroyed when stochastic layers merge; these layers widen as the deviation of the vortex position from the stationary points, increases. The efficiency of chaotization is analyzed by means of Poincaré sections and accumulated Lyapunov exponents.

1. Problem formulation

1.1. Introduction

The dynamics of vortices has been studied for over two hundred years, since vortices have been recognized as essential elements in the dynamics of turbulent and/or planetary fluid flows. Though the effects of a finite spatial extent of vortices are important when they interact strongly (i.e. when they are in a close vicinity), point vortices have been used to study many physical problems. In hydrodynamics, special attention has been given to point vortices due to the vast variety of their applications, such as the finite-amplitude evolution of jet meanders [1,2], the shedding of vortices by topographic anomalies [3–7], multi-pole dynamics [8–10]. Another application for point vortices is as components of turbulent flows [11]. Indeed, in a developed turbulent flow, distant coherent structures move as point-vortices in an unsteady external field. On the contrary, close vortices can merge if they are like-signed and if they have a finite area. Vortex merger can lead to their growth, and to the formation of filaments, which are related to energy transfers from small to large scales and to a direct enstrophy cascade, in 2D turbulent incompressible flows [12,13].

In the ocean and the atmosphere, synoptic and mesoscale vortices, often monopoles or dipoles, play an essential role in the advection and stirring of tracers. Such vortices can be identified by their Lagrangian characteristics [14,15]. When geophysical vortices are small and close to circular, point-vortices can also be used to model this advection and stirring; such studies have been carried out both for the ocean [16] and for the atmosphere [17]. Furthermore, when vortices move in the ambient fluid, they carry along fluid parcels for a long range and time. Mixing occurs on the vortex periphery. Thus heterogeneous fluid can be carried to remote regions and mixed over long distances. For instance, a Mediterranean Water tongue is found at mid-latitude in the Atlantic Ocean, between 700 and 1400 m depths, as the result of transport and mixing of this water mass by Mediterranean Water eddies (meddies) [18,19].

Recent observations have shown that oceanic vortices can come close to each other, especially in their region of formation [20].

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In the ocean, vortices are never exactly identical, and they evolve in large-scale currents (which can be approximated at first order by shear or strain flows [21]). This raises the question of the interaction of two different vortices in an external flow. The case of two identical vortices in an external flow has been previously addressed by Perrot and Carton [22,23]. They described the transition from periodicity or quasi-periodicity to chaos, when the amplitude of the external flow increases. They also investigated the modification of nonlinear regimes in the evolution of two finite-area vortices in such an external flow. In particular, they found two new regimes compared to the interaction without strain: the existence of quasi-stationary vortex configurations and vortex merger from large initial distances.

The merger of two unequal vortices has also been the subject of several studies, Melander et al. [24], Dritschel and Waugh [25], Korotaev and Dorofeev [26] and Brandt et al. [27]. These studies have classified the various nonlinear regimes in the parameter space (ratio of vortex radii, ratio of vortex strengths, normalized intercentroid distance) and they have shown that the two vortices not only can completely merge, but that a partial merger, or straining out of one vortex by the other, are possible when the two vortices are fairly asymmetric.

Nevertheless, no study has considered yet the interaction of two asymmetric vortices in an external flow, due to the multiplicity of physical parameters. Here we consider the interaction of point-vortices in a two-dimensional incompressible flow, a problem which has fewer parameters, and the chaotic advection which can occur in the vortex vicinity.

2. Model of two point vortices with arbitrary intensities in an external flow

2.1. Two-point vortex system

Let us consider two point vortices with arbitrary intensities μ_α, $\alpha = 1, 2$ in a barotropic fluid. The stream-function induced by the vortices has a simple form

$$\psi_v = \sum_{\alpha=1}^{2} \frac{\mu_\alpha}{2} \log((x - x_\alpha)^2 + (y - y_\alpha)^2), \tag{1}$$

where (x_α, y_α) are the vortex coordinates.

When the vortices are like-signed, they rotate about a center which lies between them. When the vortices are opposite-signed, they rotate about a center located outside the pair.

2.2. Adding an external deformation flow

When an external flow is superimposed on the vortices, their motion becomes considerably complicated. We consider an external deformation flow, comprising shear (S_0) and rotational (Ω_0) components, of the form

$$\psi_d = S_0(x^2 - y^2) + \Omega_0(x^2 + y^2). \tag{2}$$

The governing equations for the vortex motion have a Hamiltonian form

$$\begin{aligned} \dot{x}_\alpha &= -\left.\frac{\partial(\psi_v + \psi_d)}{\partial y}\right|_{\substack{x=x_\alpha\\ y=y_\alpha}} = 2y_\alpha(S_0 - \Omega_0) + \mu_\beta \frac{y_\beta - y_\alpha}{r_0^2}, \\ \dot{y}_\alpha &= \left.\frac{\partial(\psi_v + \psi_d)}{\partial x}\right|_{\substack{x=x_\alpha\\ y=y_\alpha}} = 2x_\alpha(S_0 + \Omega_0) - \mu_\beta \frac{x_\beta - x_\alpha}{{r_0}^2}, \end{aligned} \tag{3}$$

where $\alpha = 1, 2, \beta = 1, 2$ are the vortex indices, $\alpha \neq \beta$ and $r_0 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$- is the distance between the vortices.

Perrot and Carton [22] showed that, when $\mu_1 = \mu_2$, the dynamical system (3) with 2.5 degrees of freedom remains symmetric at all times because the center of the deformation flow coincides with the vorticity center of the vortices. Hence it can be reduced to a dynamical system with 1.5 degrees of freedom. Then, one can investigate the motion of only one vortex of the pair. Even for equal vortex intensities, the motion of the vortices can be chaotic [22]. The regularity of the vortex motion is determined by the parameters of the deformation flow oscillation and by the initial distance between the vortices r_0.

When the intensities μ_α are arbitrary, the vorticity center does not coincide with the deformation center. Thus, symmetry is broken and the vortex motion becomes more complex. As we will show below, however, it is still possible to reduce the number of degrees of freedom of the system by a unity. To reduce this dimension, we investigate the motion of the vorticity center.

The position of the vorticity center is governed by the expressions

$$x_c = \frac{L_x}{\mu_1 + \mu_2}, \qquad y_c = \frac{L_y}{\mu_1 + \mu_2}, \tag{4}$$

where

$$L_x = \mu_1 x_1 + \mu_2 x_2, \qquad L_y = \mu_1 y_1 + \mu_2 y_2 \tag{5}$$

determine the linear momentum components. Using (4) and (5), two equations can be derived for L_x and L_y

$$\begin{aligned} &\dot{L}_x = 2(S_0 - \Omega_0)L_y, \qquad \dot{L}_y = 2(S_0 + \Omega_0)L_x, \\ &\frac{\dot{L}_x}{\dot{L}_y} = \frac{(S_0 - \Omega_0)}{(S_0 + \Omega_0)} \frac{L_y}{L_x} \end{aligned} \tag{6}$$

which can be integrated in time to yield a relation between these components

$$L_x^2 = \frac{(S_0 - \Omega_0)}{(S_0 + \Omega_0)} L_y^2 + L_x^2(0) - \frac{S_0 - \Omega_0}{S_0 + \Omega_0} L_y^2(0). \tag{7}$$

Now, by integration of (6), we obtain the explicit time-dependent solution for L_x and L_y

$$\begin{aligned} \begin{pmatrix} L_x \\ L_y \end{pmatrix} &= \begin{pmatrix} L_x(0) \\ L_y(0) \end{pmatrix} \cos(\Phi(t)) \\ &\quad - \frac{\sqrt{\Omega_0^2 - S_0^2}}{S_0 + \Omega_0} \begin{pmatrix} \dfrac{L_y(0)}{S_0 + \Omega_0} \\ \dfrac{S_0 + \Omega_0}{S_0 - \Omega_0} L_x(0) \end{pmatrix} \sin(\Phi(t)), \\ \Phi(t) &= 2\sqrt{\Omega_0^2 - S_0^2}\, t. \end{aligned} \tag{8}$$

According to (8), L_x, L_y are related to a coordinate transformation and change with time. With the help of the expressions (5) and (8), the coordinates of one vortex can be expressed in the coordinates of the other one. Consequently, it allows us to decompose the system (3), yielding independent systems for each of the vortices. In other words, it allows us to reduce the number of degrees of freedom by a unity.

For the reduction of degrees of freedom of vortex systems, investigators (see for example [3]) often make use of the angular momentum, which is written as

$$M(t) = \mu_1(x_1^2 + y_1^2) + \mu_2(x_2^2 + y_2^2). \tag{9}$$

In the problem under investigation, however, the angular momentum is only conserved when the deformation center and the vorticity center coincide (i.e. when $L_x(0) = L_y(0) = 0$). Therefore in the general case of arbitrary intensities and initial positions of the vortices, the integral expression for M cannot be obtained.