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# Waves in two-layer shear flow for viscous and inviscid fluids

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ABSTRACT

A two-layer shear flow is studied for inviscid and viscous fluids. Here, the layers flow between two horizontal walls and are buoyantly stable. Each layer contains a finite amount of shear and the horizontal velocity is specified such that it is continuous when unperturbed. The interface between the two layers is given a small sinusoidal perturbation and the subsequent response of the system is studied. Different solution techniques are employed for the inviscid and viscous flows. These both rely on linearizing the governing equations for each of these flows. In particular, the viscous flow is constrained to remain within a small perturbation of the unperturbed flow as it evolves. This assumption is justified since standing wave behaviour is expected in the inviscid case. Solutions are presented for a variety of different values of the shear parameters and the way these parameter choices affect the interaction between vorticity and density in the viscous case is investigated in detail. These linearized solutions are confirmed by comparison with fully non-linear results obtained numerically.



## 1. Introduction

The presence of shear in fluid flow is associated with a variety of wavelike behaviours. Of particular interest is the effect a finite amount of shear may have on stratified flows and on fluid interfaces. Mechanisms for the generation of shear are often found in viscous flow, for instance in boundary layers or in the flow between moving plates, and similar effects are possible in rotational inviscid flow. Since shear is essentially a measure of spatial velocity gradient, it is often convenient to describe or treat these types of flows in terms of vorticity. For example, the linear velocity profile established between two moving plates (plane Couette flow) may be described as having constant vorticity. Such a description is particularly useful in dealing with wavelike behaviour or periodic disturbances, an example of which is the treatment of co-rotating vortices by Saffman [1].

Pullin and Grimshaw [2] calculated numerous large amplitude steady waves, including some with limiting features such as corners, on a two-layer inviscid flow with shear in the lower layer. Similarly, in [3] numerous steady waves were computed on a free shear layer and these featured a variety of resonant interactions between wave modes. Standing waves are often studied in the context of water waves; these are discussed in terms of their associated vorticity in [4].

A number of shear flows are unstable to small perturbations. Two examples of these are the Kelvin–Helmholtz and Holmboe instabilities. In the Kelvin–Helmholtz instability two fluid layers flow past each so that there is a thin region of infinite shear at their common boundary. This is a well known and thoroughly studied problem, and solutions for both viscous and inviscid fluids were computed in [5] where the predicted growth of the wave and the formation of a cat's-eye spiral were seen at the interface. In the Holmboe instability, the shear is spread over a layer of finite width and, as presented by Umurhan and Heifetz [6], this flow configuration permits a variety of solution modes, including travelling and standing waves. The stability of a variety of different shear flows are investigated in [7], although the focus there is on perturbing some base vorticity or velocity profiles in a few very specific ways. The flow presented in this paper will typically be perturbed by giving the interface between the layers a sinusoidal disturbance.

Two fluid layers of slightly different densities, bounded above and below by rigid walls, are considered. The lower layer is denser than the upper so that the flow is buoyantly stable. When unperturbed, each layer flows with constant vertical shear. The amount of shear in each layer may differ, but the associated horizontal velocity is chosen so that it is continuous across the interface between the fluids. There are two cases here that are of interest: the case where both layers flow with equal amounts of shear and the case where one layer has no shear.

Two versions of this flow will be considered: one that assumes both fluid layers are inviscid and one that includes the effects of viscosity. The inviscid version will be based on a classical description of a two-layer incompressible fluid, with an infinitely thin interface separating the two layers. By contrast, the viscous version of the flow will feature a continuously stratified, weakly

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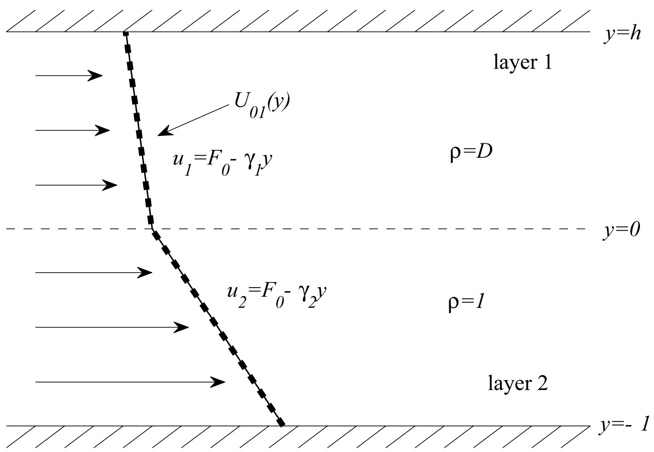


**Fig. 1.** Schematic of the two-layer shear flow. The heavy dashed line shows the continuous horizontal velocity profile $U_{01}(y)$ used in the viscous formulation, while the thin solid line is the horizontal velocity profile for the inviscid case.

compressible fluid, albeit one that mimics the layered fluid of the inviscid case. The interface (or quasi-interface) in each of these two flows will be given a small sinusoidal perturbation (thus perturbing both density and velocity) and the subsequent behaviour of the wave in the interfacial region will be the focus of this work.

The two different formulations of the flow are introduced in Section 2 and care has been taken to ensure that these match each other as closely as possible. This enables direct comparison so that the effects of viscosity may be assessed as in [8]. Both formulations are then studied using linearization techniques; the assumption here is that, once perturbed, the evolving flow does not change too far from the base flow. In Section 3, this is achieved by approximating the fully non-linear governing equations of the inviscid problem with their linearized equivalents. Similarly, the viscous formulation is carefully analysed using perturbation series techniques and a spectral solution is specified in Section 4. Here much attention is given to choosing appropriate initial conditions so that the system is perturbed in a similar way to the inviscid problem. The results of these two solution techniques are compared in Section 5, where a variety of solutions for different parameter values are presented. Notably these display a range of oscillatory behaviours, including standing waves, both damped and undamped. The validity of these solutions in the context of the solution technique for the viscous flow will be discussed.

## 2. Model formulation

The flow to be considered consists of two horizontal fluid layers of different densities. The upper fluid is denoted as layer 1 and the lower fluid as layer 2; quantities associated with each layer are subscripted accordingly. These layers are in motion with a continuous horizontal velocity profile, such that the speed at the interface of the two layers is $c_0$ and each layer flows with constant vertical shear, namely $\omega_1$ in the top layer and $\omega_2$ in the lower layer; hence the base horizontal speeds of each layer are $u_1 = c_0 - \omega_1 y$ and $u_2 = c_0 - \omega_2 y$, respectively. There is thus a sharp change in both density and vorticity about $y = 0$, although the horizontal speed is continuous here. Walls are present above and below the interface, at $y = h_1$ and $y = -h_2$, respectively. A schematic diagram of the flow configuration is shown in Fig. 1.

Non-dimensional variables will be introduced for convenience. The length scale is chosen to be the depth of the lower layer $h_2$. It follows that an appropriate choice for the speed scale is $\sqrt{gh_2}$ and similarly the time scale to be used is $\sqrt{h_2/g}$. The lower layer density $\rho_2$ is used to scale density. This gives a number of key dimensionless parameters, namely a Froude number $F_0 = c_0/\sqrt{gh_2}$, two dimensionless measures of shear $\gamma_1 = \omega_1\sqrt{h_2/g}$ and $\gamma_2 = \omega_2\sqrt{h_2/g}$, a density ratio $D = \rho_1/\rho_2$ and the dimensionless height of the upper layer $h = h_1/h_2$. This set of five parameters will be used for both the inviscid and viscous formulations presented below.

There will be a few key choices for these parameters. In particular, interest lies in investigating the effect of changing the strength of the shear parameters $\gamma_1$ and $\gamma_2$ in each layer. The case of equal shear, that is where $\gamma_1 = \gamma_2$, will be studied first. The stability of a similar flow with a continuously stratified density profile (and only the lower wall) was examined by Chandrasekhar [9, article 103a]. In viscous fluids, shear flows of this type are often referred to as plane Couette flow (see, for instance, Drazin and Reid [10, Chapters 4 and 5] for various approaches to the viscous problem or Case [11] for an investigation of the stability of the equivalent inviscid flow), namely the flow induced between moving plates. The next step is to consider the related case of $\gamma_1 = 0$ and $\gamma_2 \neq 0$. A similar flow was studied by Pullin and Grimshaw [2] where numerous large amplitude steady waves (including over hanging waves) were computed and it is possible that steady waves of a smaller amplitude may be obtained here. In each of these cases the base flow will be given a small sinusoidal perturbation in both density and vorticity. It is the response to this disturbance and subsequent evolution of the flow, with particular emphasis on the interfacial region, that will be the focus of the study.

### 2.1. Inviscid formulation

The inviscid version of this problem involves two immiscible fluid layers flowing as described above. Both layers are assumed to be inviscid and incompressible. There is an interface between the layers lying at $y = 0$ when the system is unperturbed, and more generally the interface is represented by the function $y = \eta(x, t)$. This implies that the exact shape of the layers is not known a priori, and by the very nature of the problem the layers' shapes change as the interface evolves.

The inclusion of shear means that the flow is inherently rotational; however, as only constant shear is considered it is possible to write the velocity as a sum of rotational and irrotational parts, thus allowing velocity potentials $\Phi_1$ and $\Phi_2$ to be constructed for the irrotational parts of the fluid motions in each layer. In the upper layer, between $y = \eta(x, t)$ and $y = h$, the velocity components are

$$u_1 = F_0 - \gamma_1 y + \frac{\partial \Phi_1}{\partial x}$$

$$v_1 = \frac{\partial \Phi_1}{\partial y}$$

and similarly in the lower layer, between $y = -1$ and $y = \eta(x, t)$, the horizontal and vertical components of velocity are written as

$$u_2 = F_0 - \gamma_2 y + \frac{\partial \Phi_2}{\partial x}$$

$$v_2 = \frac{\partial \Phi_2}{\partial y}.$$

The velocity potentials satisfy Laplace's equation in their respective regions, that is

$$\nabla^2 \Phi_1 = 0 \quad \eta(x, t) < y < h$$

$$\nabla^2 \Phi_2 = 0 \quad -1 < y < \eta(x, t)$$

as is usual for incompressible inviscid fluids. Additionally, there are a number of boundary conditions to be defined on the interface. On either side of the interface it is required that the normal component of velocity is zero, leading to the condition that

$$\frac{\partial \eta}{\partial t} = v_i - u_i \frac{\partial \eta}{\partial x} \quad \text{on } y = \eta(x, t) \tag{2.1}$$