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Aligned and nonaligned radial stagnation flow on a stretching cylinder

Patrick D. Weidman [a,*], Mohamed E. Ali [b]

[a] Department of Mechanical Engineering, University of Colorado, Boulder, CO 80309-0427, USA

[b] King Saud University, Department of Mechanical Engineering, P.O. 800, Riyadh, 11421, Saudia Arabia

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ABSTRACT

Laminar radial stagnation flow impinging on a stretching or shrinking elastic cylinder of radius a is studied. The strain rate of the stagnation flow is $2k$ and that of the stretching cylinder is b. The origin of stretching is in general displaced by a distance c from the inviscid stagnation circle on the cylinder. An exact similarity reduction of the Navier–Stokes equations leads to coupled ordinary differential equations describing the primary flow $f(\eta)$ and a secondary flow $g(\eta)$ with similarity variable $\eta = (r/a)^2$. The system is governed by the Reynolds number $R = ka^2/2\nu$, the dimensionless offset parameter $\alpha = c/a$, and the dimensionless stretching parameter $\beta = b/2k$, where ν is the kinematic viscosity of the fluid. Solutions of the coupled equations only depend on R and β, but the flow field depends crucially on α. Analytic solutions are found for the special values $R = 2+\beta$ and also for all β if $R = 1$. For other values of R and β, solutions are obtained numerically. We find no solutions for $\beta < \beta_c$, dual solutions when $\beta_c \leq \beta < -1$, and unique solutions for $\beta > -1$, where β_c depends on R. The stability of the dual primary flow solutions is determined and the effect of flow misalignment is displayed in streamfunction plots.

1. Introduction

In the manufacture of metal and polymer solid cylinders, the material is usually in a molten phase when thrust through an extrusion die and then cools and solidifies some distance away from the die. Experiments by Vleggaar [1] show that the velocity of the material is approximately proportional to the distance, so these systems are often modeled as linearly stretching rods or cylinders. For metals and polymers, and especially when the material is rubber or plastic, it is advantageous to blow a gaseous medium onto the not-yet-cooled material to provide quality control in the cooling process [2], in which case the stagnation circle of the gaseous stream may coincide with, or lie downstream of, the origin of stretching (die exit). In the event that the blowing stream can be modeled as normal radial stagnation flow on the extruding cylinder, we have the situation of interest in this paper: aligned or nonaligned radial stagnation flow on a stretching cylinder.

Stagnation-point flows are ubiquitous in the sense that they appear in virtually all flow fields of engineering and scientific interest. In many situations, it is possible to find an exact symmetric stagnation-point flow solution of the Navier–Stokes equations; see, for example, the review by Wang [3]. Exact asymmetric stagnation-point solutions, on the other hand, are rare. While symmetric exact solutions based on a unique strain rate were reported by Heimenz [4] for planar (two-dimensional) and by Homann [5] for axisymmetric (three-dimensional) stagnation-point flows, Howarth [6] showed that an exact non-axisymmetric solution based on two orthogonal strain rates is available. Wang [7] and later Cox [8] applied Howarth's reduction to the problem of flow between an air bearing table and a floating disk to obtain another exact nonaxisymmetric solution of the Navier–Stokes equations; Cox [8] concluded that the aerodynamic lift on the disk could be enhanced if the impinging flow were not axisymmetric. Recently, Wang [9] devised a similarity reduction of the Navier–Stokes equations which accounts for the misalignment between an external stagnation flow and a stretching sheet; both Hiemenz [4] and Homann [5] stagnation-point flows impinging, respectively, on planar and axisymmetrically stretching sheets were considered.

In a seminal paper, Wang [10] reported a similarity reduction of the Navier–Stokes equations describing axisymmetric radial stagnation flow normal to a circular cylinder. Many extensions of this work to Newtonian and non-Newtonian fluids have since been reported. The symmetry of the radial stagnation flow is destroyed if the cylinder is allowed to move axially, as shown by Gorla [11] for the case of constant cylinder velocity. The influence of cylinder rotation and wall transpiration in radial stagnation flows was reported by Cunning, et al. [12]. A formulation for oblique stagnation flow on a circular cylinder was reported by Okamoto [13] and later rediscovered by Weidman and Putkaradze [14]. Flows interior and exterior to a stretching cylinder were studied by Brady and Acrivos [15] and Wang [16], respectively; these flows are reflexively symmetric about the stretching origin and axisymmetric

* Corresponding author. Tel.: +1 303 492 4684; fax: +1 303 492 3498.
E-mail address: weidman@colorado.edu (P.D. Weidman).

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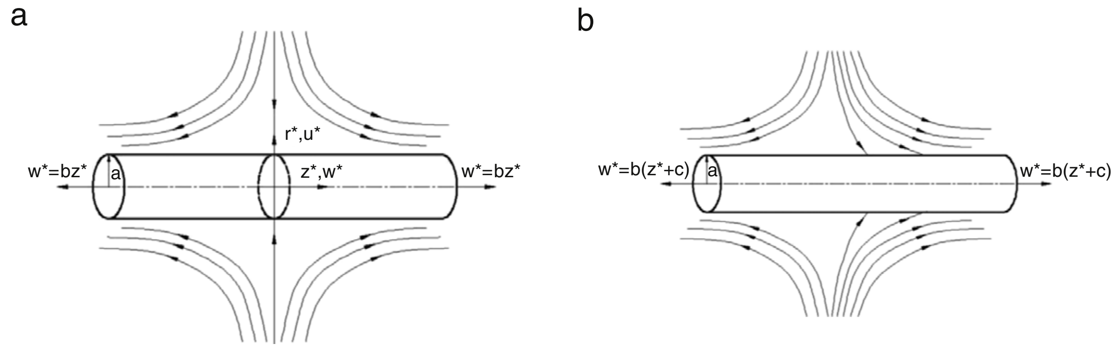

Fig. 1. (a) Coordinates and sketch of the aligned streamline pattern that appears for $c = 0$ and (b) sketch of the nonaligned streamline pattern that appears for $c \neq 0$.

about the cylinder axis. If one superposes radial stagnation flow onto a cylinder stretching about the impingement point of inviscid stagnation flow, the streamlines will always be axisymmetric. However, the reflexive symmetry that exists when the stagnation circle coincides with the circle of stretching will be lost when the two circles are axially displaced. In the present investigation, we show how both symmetric and asymmetric exact Navier–Stokes solutions can be attained for this problem. Calculations over the parameter space of solutions is an order of magnitude more time consuming than for the planar problem studied by Wang [9] owing to the additional parameter R governing the flow curvature. Our goal is to fully understand the kinematical features of the asymmetric flow, to determine the wall shear stress as a function of the governing parameters, and to ascertain the stability of the primary flow on which the secondary flow depends. At the same time, we provide some exact solutions that exist for special values of the governing parameters.

The presentation begins in Section 2 with the problem formulation, and Section 3 outlines the kinematical features of the flow with particular reference to the misaligned case. The analytical solutions given in Section 4 are followed by a presentation of relevant numerical results in Section 5. A self-similar stability analysis to determine which of the dual solutions are stable is given in Section 6 and the paper ends with a summary and conclusion in Section 7, followed by an Appendix giving some exact results for the planar problem.

2. Problem formulation

We consider isothermal, normal radial stagnation flow on a linearly stretching/shrinking elastic cylinder. We further allow the origin of stretching to be axially displaced from the inviscid stagnation flow circle. Variables with an asterisk are dimensional and those without an asterisk are dimensionless. For axisymmetric flow in the absence of swirl, we take (u^*, w^*) as velocities in the (r^*, z^*) coordinate directions and denote the pressure by p^*. The steady flow is governed by the equation of continuity

$$\nabla \cdot \mathbf{u}^* = 0 \tag{2.1a}$$

and the viscous incompressible Navier–Stokes equation

$$(\mathbf{u}^* \cdot \nabla)\mathbf{u}^* = -\frac{1}{\rho}\nabla p^* + \nu \nabla^2 \mathbf{u}^*, \tag{2.1b}$$

where ρ and ν are the fluid density and kinematic viscosity, respectively, here assumed constant. As $r^* \to \infty$, the viscous flow approaches the potential stagnation flow

$$u^* = -k\left(r^* - \frac{a^2}{r^*}\right), \qquad w^* = 2kz^* \tag{2.2a}$$

$$p^* = p_0^* - \frac{\rho k^2}{2}\left[\left(r^* - \frac{a^2}{r^*}\right)\right], \tag{2.2b}$$

in which a is the cylinder radius and p_0^* is the pressure on the stagnation circle at $z^* = 0$. Furthermore, the cylinder stretches in its own plane with strain rate b,

$$w^*(a, z^*) = b(z^* + c), \tag{2.3}$$

with the stretching origin shifted a distance $z^* = -c$ from the inviscid stagnation circle. A sketch of the coordinate system and streamlines for aligned flow with $c = 0$ is given in Fig. 1(a); when $c \neq 0$, the flow will be nonaligned, with asymmetric streamlines, as sketched in Fig. 1(b).

Introducing the dimensionless coordinates

$$r = \frac{r^*}{a}, \qquad z = \frac{z^*}{a}, \qquad \eta = r^2, \tag{2.4}$$

we find a reduction of the Navier–Stokes equations using the following coordinate separation of the velocity field:

$$\begin{aligned} &u^*(\eta) = -k\,a\frac{f(\eta)}{\eta^{1/2}}, \\ &w^*(\eta, z) = 2ka[zf'(\eta) + \alpha\beta g'(\eta)], \qquad p^* = \rho k^2 a^2 P \end{aligned} \quad , \tag{2.5}$$

where $\alpha = c/a$ is the dimensionless offset parameter and $\beta = b/2k$ is the ratio of the strain rates. This ansatz satisfies the continuity equation (2.1a), and insertion into the incompressible Navier–Stokes equation (2.1b) yields, when far-field matching with (2.2a) is taken into account, the coupled pair of ordinary differential equations and an expression for the pressure

$$\eta f''' + f'' + R(ff'' - f'^2 + 1) = 0 \tag{2.6a}$$

$$\eta g''' + g'' + R(fg'' - f'g') = 0 \tag{2.6b}$$

$$P = P_0 - \left[\left(\frac{f^2}{2\eta}\right) + 2k^2z^2 + \frac{1}{R}f'(\eta)\right], \tag{2.6c}$$

where $R = ka^2/2\nu$ is the Reynolds number introduced by Wang [10]. The boundary conditions are that the cylinder surface is impermeable and satisfies the no-slip boundary condition. The solution should tend towards a displaced form of the inviscid solution (2.2) as $\eta \to \infty$. Thus Eqs. (2.6a) and (2.6b) must satisfy the boundary conditions

$$f(1) = 0, \qquad f'(1) = \beta, \qquad f'(\infty) = 1 \tag{2.7a}$$

$$g(1) = 0, \qquad g'(1) = 1, \qquad g'(\infty) = 0. \tag{2.7b}$$

Notice in (2.6a) and (2.6b) the one-way coupling similar to [9], whereby the function $f(\eta)$ influences $g(\eta)$, but not vice versa.

The boundary condition on $g(1)$ is chosen to ensure that the dimensionless streamfunction

$$\psi(\eta, z) = \frac{\psi^*(r^*, z^*)}{ka^3} = zf(\eta) + \alpha\beta g(\eta) \tag{2.8a}$$

is zero on the surface of the cylinder. The meridional velocities are obtained using the relations