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Numerical analysis of a weighted-residual integral boundary-layer model for nonlinear dynamics of falling liquid films

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Abstract

The nonlinear dynamics of thin liquid films falling on a vertical plane is investigated numerically using the first-order time-dependent weighted-residual integral boundary layer (WRIBL) equations derived by Ruyer-Quil and Manneville (2000). We validate the WRIBL equations by comparison of its solutions with those of its second-order version, solutions obtained by both stationary and time-dependent direct numerical simulation and experiments. We find that sufficiently close to the stability threshold of the system with periodic boundary conditions, the emerging waves are of γ_1-type. However, beyond a secondary bifurcation threshold, γ_2-type waves emerge and can coexist with γ_1-waves. The analysis of the WRIBL equations reveals, similar to the first-order Benney equation (BE), the existence of both periodic traveling wave (TW) and aperiodic non-stationary wave (NSW) flows. It is shown that although the WRIBL equations display bounded solutions for significantly larger values of the Reynolds number than the BE, they may exhibit negative flow rate which consists of reverse flow against gravity. The threshold for emergence of these solutions is a zero local flow rate that appears, for sufficiently large Kapitza numbers, to correspond to a specific value of the normalized wave height.

Keywords: Falling liquid films; Longwave theory; Nonlinear dynamics; Bifurcations

1. Introduction

Falling liquid films are often encountered in various technological applications, such as evaporators, condensers, heat exchangers, coating, and physical phenomena, such as gravity currents and lava flows. Significant progress has been attained in the analysis of thin (macroscopic) liquid films. Oron et al. [1] unified such analyses into a comprehensive framework in which the special cases naturally emerged. Employing the long-wave approximation Oron et al. [1] derived a generic evolution equation describing the spatio-temporal dynamics of a liquid film subjected to various physical mechanisms.

The long-wave approach has also been found to be a useful tool of investigation in the case when the base state is a flow and the Reynolds number of the flow is *not* large. An example is the dynamics of a liquid film flow on a vertical or inclined plane, where the steady Nusselt flow is known to be unstable to small long-wave disturbances.

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The Nusselt flow undergoes a Hopf bifurcation which evolves to temporal periodic or aperiodic waves, and the final selection depends on the flow parameter set and possibly the initial condition. In his pioneering work, Benney [2] derived the evolution equation referred nowadays as to the Benney equation (BE). This evolution equation describes the nonlinear dynamics of the interface of a two-dimensional liquid film flowing on a fixed inclined plane. The Benney equation in its dimensionless form can be written [3]

$$h_t + \frac{2}{3}(h^3)_x + \varepsilon\left[\left(\frac{8R}{15}h^6 - \cot\theta h^3\right)h_x + Wh^3 h_{xxx}\right]_x = 0, \tag{1}$$

where $R = gd^3/\nu^2 \sin\theta$ is the unit-order Reynolds number of the flow driven by gravity g, $W = 2\varepsilon^2\sigma/(3\rho g d^2)$ is the rescaled inverse capillary number related to surface tension σ, θ is the angle of plane inclination with respect to the horizontal, ρ is the fluid density, d is the average film thickness, ν is the kinematic viscosity of the liquid, and ε represents the ratio between d and the wavelength of the characteristic interfacial disturbance λ. The independent variables x and t are rescaled non-dimensional spatial and temporal variables, respectively, and $h = h(x, t)$ is the local dimensionless thickness of the film scaled with d.

The Benney equation has been extensively studied over the years. Gjevik [3] introduced an approximate modal approach for a study of spatially periodic solutions of Eq. (1) Nakaya [4] constructed various traveling wave solutions for Eq. (1) in the infinite and periodic domains and investigated their properties. Salamon et al. [5] carried out the study of traveling waves on vertical films by solving the full *stationary*, i.e., steady in the moving frame of reference and describing traveling waves, hydrodynamic equations directly along with the free-surface boundary conditions using the finite element method. They also compared between some of these solutions with the traveling wave solutions of Eq. (1) and found a good agreement in certain domains. Direct numerical simulations of the Navier–Stokes equations for falling vertical liquid films were carried out by several groups [6–12]. Joo and Davis [13] showed in the BE that all two-dimensional saturated waves are unstable to two-dimensional spatially subharmonic disturbances in the downstream direction culminating in apparently chaotic dynamics. They also investigated [14] the three-dimensional version of the BE to find complex three-dimensional cross-stream patterns. Ramaswamy et al. [15] were the first to document a quasiperiodic wave regime in an exhaustive numerical investigation of the time-dependent Navier–Stokes equations for falling films. Gao et al. [7] recently obtained quasiperiodic solutions for the same parameters as employed by Ramaswamy et al. [15] using the volume of fluid numerical technique. However, the wave amplitude as obtained in their computations is significantly lower than in computations made in [15]. Oron and Gottlieb [16] solved the BE numerically and discovered that for certain parameter conditions the BE spontaneously exhibited aperiodic non-stationary waves for standard initial data. These non-stationary waves were found to correspond to multimode interactions which were later analyzed by Gottlieb and Oron [17] in a fourth-order complex-mode projection to yield simple quasiperiodic waves in a reduced eighth-order dynamical system. Gottlieb and Oron [17] also discovered coexisting traveling waves and non-stationary waves in their complex-mode projection of the BE. However, no evidence of coexisting solutions was found in the corresponding numerical investigation of the BE. We note that determination of coexisting attractors in the boundary value problems for partial differential equations of the evolution type involves an extremely difficult search in the infinite-dimensional functional space of spatially-dependent initial conditions.

However, along with the success of the Benney equation model to describe the dynamics of falling liquid films, there is a serious drawback. It was found that the solutions of the Benney equation grow without bound in a certain subdomain of the linearly unstable region of the system. In this case the BE loses its physical relevance. The feature of solutions blow-up for the Benney equation was first noticed by Pumir et al. [18] and further studied by Rosenau et al. [19] and Oron and Gottlieb [16], where the valid range of the Benney equation was mapped in the appropriate parameter space. Gottlieb and Oron [17] investigated the bifurcation structure near the blow-up and determined that it corresponds to a saddle-node bifurcation. This structure was later verified by Scheid et al. [20] in the analysis of a corresponding third-order ordinary differential equation deduced for an assumed traveling wave form. The primary bifurcation of the first- and second-order Benney equations was investigated by Lin [21] and Oron and Gottlieb [22], who also studied the issue of sideband instability of monochromatic traveling waves. Oron and Gottlieb [22] demonstrated that the asymptotic series leading to the derivation of the Benney equation may be poorly converging.

To overcome some of the drawbacks associated with the BE for fluids with large surface tension (large Kapitza number) several approaches were recently attempted. Ooshida [23] proposed a regularization of the BE based on Pade approximations to the flux term. An alternative approach was introduced by Ruyer-Quil and Manneville [24,25] extending the boundary-layer theory developed by Shkadov [26]. The Shkadov theory was shown [27,28] to be suc-