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The unsteady MHD boundary-layer flow on a shrinking sheet

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ABSTRACT

The unsteady boundary-layer flow on a shrinking surface in an electrically conducting fluid is considered as it develops from rest. The nature of the solution is shown to depend on a dimensionless magnetic parameter M. For $M > 1$ a steady state is reached at large times, when $M = 1$ there is also a boundary-layer flow for all times but now with a thickness growing at a rate proportional to t (dimensionless time). However, for $M < 1$ the solution breaks down at a finite time t_s with the boundary-layer thickness and maximum velocity becoming large as t approaches t_s, though with the skin friction remaining finite.

1. Introduction

The viscous flow past a moving surface initiated by Sakiadis [1] has since received a considerable amount of attention because of its engineering applications. In particular, the case of the boundary-layer flow over a stretching sheet has attracted much research due, in part, to its mathematical simplicity and also because closed form solutions are possible in many cases (see, for example, [2–4]), starting with the work by Crane [5]. More recently, the solution given by Crane has been generalized for an arbitrary stretching sheet with a suitable transpiration velocity by Weidman and Magyari [6]. The generalized stretching velocity also includes an important class of backward boundary layer flows (see, for example, [7,8]). As pointed out by Wang [9] solutions do not exist for a shrinking sheet in an otherwise still fluid and, in order to have a solution, Micklavcic and Wang [10] added constant suction. Wang [9] treated a stagnation-point flow and Fang and Zhang [11] added a transverse magnetic field to contain the vorticity over a shrinking sheet. The effects of mass transfer and heat transfer on flow generated by a shrinking sheet have been considered by Fang et al. [12] and Fang and Zhang [13].

The problem of an unsteady boundary-layer flow past stretching surfaces has received much less attention for two basic reasons, namely these problems are much more difficult to analyse than the corresponding steady state problems and secondly steady state problems are, perhaps, of more practical relevance. Pop and Na [14] obtained a small time expansion for the problem of flow past an impulsively stretching sheet. There have been very few studies, for example Riley [15] and Harris et al. [16], which consider analytically both the initial unsteady development and the approach to the final steady state. There have been some studies which have considered only the initial flow field analytically. Nazar et al. [17] studied the unsteady flow field caused by impulsively stretching the surface and creating motion in the free stream. Most of the studies, for example Kumaran et al. [18], do not consider both the initial development and final approach analytically. The unsteadiness in the problem considered by Kumaran et al. [18] is due to sudden step change in constant transverse magnetic field where the initial state is an already established steady flow.

The present problem is interesting and new in two ways. It is a backward unsteady problem and the nature of how the flow develops from its initial state depends on the (dimensionless) magnetic parameter M. For $M > 1$ a steady state is reached at large times, when $M = 1$ there is also a boundary-layer flow for all times but now with a thickness growing at a rate proportional to t (dimensionless time). However, for $M < 1$ the solution breaks down at a finite time t_s with the boundary-layer thickness and maximum velocity becoming large as t approaches t_s, though with the skin friction remaining finite. This singularity as $t \to t_s$ is also seen in the $M = 0$ case, a situation that has some similarities with the flow development near a rear stagnation point treated by Proudman and Johnson [19] and Robbins and Howarth [20], though in this latter case a solution was seen to exist for all $t > 0$ with the boundary-layer thickness increasing exponentially over time.

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We start by deriving our model and then consider the time development of the solution in the three separate cases, $M > 1$, $M = 1$ and $M < 1$.

2. Model

We consider an unsteady two-dimensional laminar incompressible boundary-layer flow over an impulsively shrinking sheet in an electrically conducting fluid under a constant transverse magnetic field of strength B_0. The x'-axis is taken along the shrinking surface in a direction opposite to the motion of the surface and the y'-axis is taken normal to this surface (in the direction of the magnetic field). u' and v' are respectively the velocity components in the x' and y' directions. The boundary-layer equations governing the flow are then (see [18] for example),

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{1}$$

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\left(\frac{\sigma B_0^2}{\rho}\right)u' + \nu\frac{\partial^2 u'}{\partial y'^2} \tag{2}$$

where t' is time and ν, ρ and σ are respectively the kinematic viscosity, density and the magnetic permeability of the fluid. We assume that the surface is at rest for $t' < 0$ with the motion being set up for $t' > 0$ by the surface being set into motion with a velocity $U_w = -C_0 x'$ along its length, where C_0 is a positive constant. This leads to the boundary conditions

$$u' = 0, \qquad v' = 0 \quad \text{for } x' \geq 0, y' \geq 0 \ (t' < 0) \tag{3}$$

$$u' = U_w, \qquad v' = 0 \quad \text{on } y' = 0,$$

$$u' \to 0 \quad \text{as } y' \to \infty \ (x' \geq 0, t' > 0). \tag{4}$$

We make Eqs. (1)–(4) dimensionless by introducing the variables

$$(x, y) = (x', y')\left(\frac{C_0}{\nu}\right)^{1/2}, \qquad t = C_0 t',$$
$$(u', v') = (C_0\nu)^{1/2}(u, v). \tag{5}$$

We then introduce a dimensionless stream function ψ, with $u = \psi_y$, $v = -\psi_x$, so as to satisfy the continuity equation. Then to describe the flow near the stagnation point associated with the surface velocity given in (3) we put, ignoring any leading edge effects,

$$\psi(x, y, t) = -xf(y, t). \tag{6}$$

This results in

$$\frac{\partial^3 f}{\partial y^3} - f\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2 - M\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y \partial t} \tag{7}$$

on $0 \leq y < \infty, t > 0$ subject to the initial and boundary conditions that

$$f = 0 \quad \text{at } t = 0 \ (0 \leq y < \infty) \tag{8}$$

$$f = 0, \qquad \frac{\partial f}{\partial y} = 1 \quad \text{on } y = 0,$$

$$\frac{\partial f}{\partial y} \to 0 \quad \text{as } y \to \infty \ (t > 0) \tag{9}$$

where $M = \frac{\sigma B_0^2}{C_0 \rho}$ is the magnetic number.

We are concerned with the time development of the solution to Eq. (7) starting with condition (8). There are three cases to consider depending on whether $M > 1$, $M = 1$ or $M < 1$.

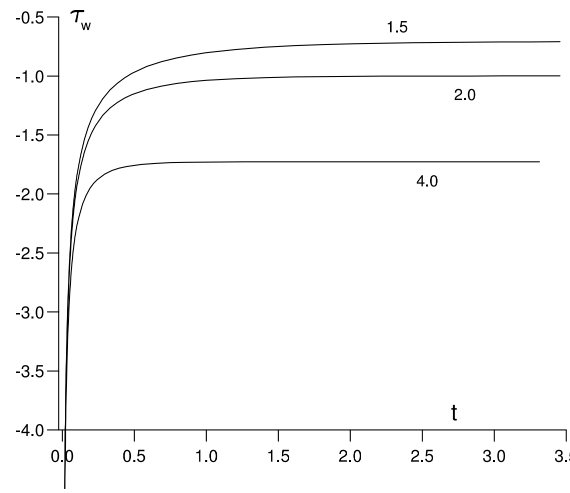

Fig. 1. Values of $\tau_w = \left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$ obtained from the numerical solution of Eqs. (7)–(9) for $M = 1.5, 2.0, 4.0$, showing the solution rapidly approaching the large-time asymptotic limit (11).

3. Solution

In all cases the initial behaviour of the solution to Eqs. (7)–(9) is purely diffusional, developing over a length scale of $y \sim t^{1/2}$ for t small, giving

$$\tau_w \sim -\frac{t^{-1/2}}{\sqrt{\pi}} + \cdots, \qquad f_\infty \sim \frac{2}{\sqrt{\pi}}t^{1/2} + \cdots \quad \text{for } t \text{ small} \tag{10}$$

where $\tau_w = \left(\frac{\partial^2 f}{\partial y^2}\right)_{y=0}$ is the wall stress and $f_\infty = \lim_{y\to\infty} f(y, t)$ is the entrainment velocity. However, the way the solution develops as t increases from these small times depends on the value of M and we start by considering the case $M > 1$.

3.1. Case $M > 1$

Eq. (7) subject to initial and boundary conditions (8), (9) was solved numerically using a method used extensively before, see [21,22] for example. This method is based on the Crank–Nicolson scheme with Newton–Raphson iteration being used to solve the resulting nonlinear finite-difference equations at each time step. An accuracy check was built into the numerical scheme allowing a variable time step Δt to be used. This meant that Δt could be increased up to some preset upper bound as the solution approached a steady state for $M > 1$ and decreased to successively smaller values of Δt as the singularity was approached for $M < 1$.

In this case our numerical integrations of (7)–(9) indicate that the solution approaches a steady state as $t \to \infty$, with

$$f(y, t) \to \frac{1}{\sqrt{M-1}}\left(1 - e^{-\sqrt{M-1}y}\right) \quad \text{as } t \to \infty. \tag{11}$$

We illustrate this in Fig. 1 with plots of τ_w against t for representative values of M. From (10) τ_w is large and negative for t small, also being independent of M in this limit, as can be seen from the plots in Fig. 1 for small times. As t increases the large-time asymptotic limit (11) is rapidly approached, with this being more quickly approached as M is increased. This behaviour was also seen for the other values of $M > 1$ attempted.

We can acquire an insight into the nature of the problem in this case by looking for a solution for M large. Expression (11) suggests that we put

$$f = M^{-1/2}\tilde{f}, \qquad \tilde{y} = M^{1/2}y, \qquad \tilde{t} = Mt. \tag{12}$$