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# Free surface deformation due to a source and a sink of equal strength in Stokes flow

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ABSTRACT

Two-dimensional Stokes flow due to a source and a sink of equal strength below the free surface is analyzed and free surface shape and cusp formation are discussed. The source-sink pair below the free surface are aligned vertical to the free surface. In the analysis, the Stokes' approximation is used and surface tension effects are included, but gravity is neglected. The solution is obtained by using conformal mapping and complex function theory. From the solution, typical free surface shapes are shown and formation of a cusp on the free surface is discussed. As the capillary number increases, the converging free surface shape becomes singular and tends to form a cusp for sufficiently large capillary number. Typically, streamline patterns for some capillary numbers are also shown. As the capillary number vanishes, the solution is reduced to the linearized potential flow solution.



## 1. Introduction

Since some flow visualizations by photographs of the cusp on the free-surface of both Newtonian and non-Newtonian fluids at low Reynolds numbers have been performed by Joseph [1], many theoretical and experimental researches are carried out [2,3]. These photographs show compelling evidence for the formation of two-dimensional sharp cusps on the free surface in the regions of converging flow. The dynamical mechanism related to this cusp singularity on the free surface in a Stokes flow has recently become an attractive subject of theoretical and experimental researches. The understanding of cusp formation is very important, since the presence of two-dimensional cusp on the free surface may result in a mechanism for air entrainment such as in chemical reaction or film coating. In the paper of Jeong and Moffatt [3], they carried out experiments using a pair of counter-rotating cylinders in a Newtonian fluid with free surface. For very slow rotation rates, there is a stagnation line on the free surface, and in some circumstances a small rounded crest can form in the neighborhood of this stagnation line. When the rotation rate is increased however, the surface dips downward, and simple visual observation indicates the presence of a very sharp cusp on the free surface. A complete analytical solution of a model problem of this experiments was also obtained, where the full nature of the flow and the detail of the cusp formation procedure were explained. In the model problem, they considered a Stokes flow induced by a vortex dipole below the otherwise undisturbed free surface, where direction of the vortex dipole was perpendicular to the free surface. Jeong [4] generalized this model problem to the case in which the orientation of the vortex dipole beneath the free surface is arbitrary. Antanovskii [5] generalized to the case in which the interfacial tension is variable, and Cummings [6] to the steady bubble solutions in dipole-driven Stokes flow. Jeong [7] also considered a Stokes flow induced by a single source or sink of arbitrary strength instead of a vortex dipole below the otherwise undisturbed free surface.

In this paper, we consider a Stokes flow due to a source and a sink of equal strength located below the free surface. The source-sink pair is aligned vertical to the free surface and the distance between the source and the sink is arbitrary. This flow is a generalized flow of Jeong and Moffatt [3] and Jeong [7], since these problems may be considered as the limit cases where the distance between the source and the sink approaches to zero and infinity, respectively. As a low Reynolds number limit, the Stokes' approximation is used and the effect of surface tension is included, but gravity effects are neglected. The solution is obtained analytically by using conformal mapping and complex function theory. From the solution, the deformation of the free surface and the formation of a cusp on the free surface are discussed and some streamline patterns are shown.

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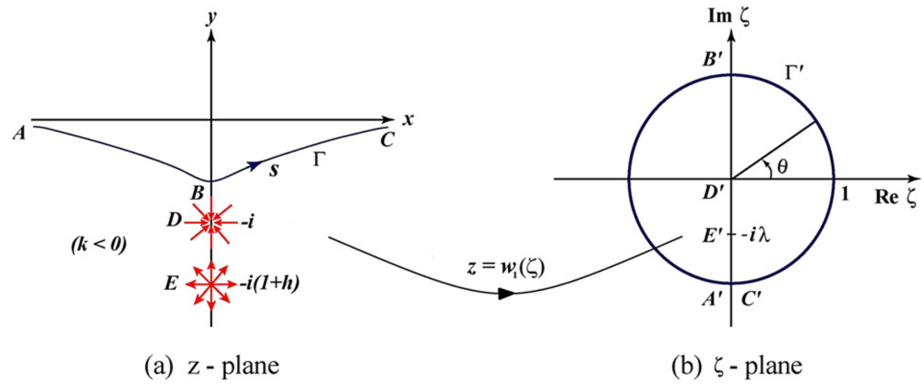


**Fig. 1.** Free surface deformation by a source-sink pair and conformal mapping from the flow field in $z$-plane into the inside of unit circle in $\zeta$-plane.

## 2. The method of solution

### 2.1. Mathematical formulation

Consider the flow region shown in Fig. 1(a), where the undisturbed fluid occupies the half space $y < 0$. A line source of strength $k$ (outward flowrate $2\pi k$) is located at $z(= x + iy) = -id$ and of strength $-k$ at $z = -id(1 + h)$. Source of negative strength may be considered as sink. To non-dimensionalize the length scale by $d$, we may take $d = 1$ in what follows and the source and the sink singularities are then located at $z = -i$ and $z = -i(1 + h)$, respectively. The flow is generated by this source-sink pair and the free surface $\Gamma$ is distorted, while the free surface at far field should remain flat and the height of the free surface vanishes as $|x| \to \infty$.

Assume that the Reynolds number $Re \equiv k/\nu$ ($\nu$: kinematic viscosity) is small so that the stream function $\Psi$ satisfies the biharmonic equation $\nabla^4\Psi = 0$. It is well known that the stream function $\Psi$ can then be expressed in the form [8]

$$\Psi = \mathrm{Im}\big[f(z) + \bar{z}g(z)\big], \tag{1}$$

where two complex functions $f(z)$, $g(z)$ are analytic at all points $z$ in the fluid domain except at the source and sink singularities $z = -i$ and $z = -i(1 + h)$, i.e.,

$$f(z) \to k\ln(z + i) \quad \text{as } z \to -i, \tag{2}$$

$$f(z) \to -k\ln\big\{z + i(1 + h)\big\} \quad \text{as } z \to -i(1 + h). \tag{3}$$

The velocity components $(u, v)$ are then given by

$$u - iv = f'(z) + \bar{z}g'(z) - \overline{g(z)}, \tag{4}$$

and the pressure $(p)$ and vorticity $(\Omega)$ fields are given by

$$p - i\mu\Omega = 4\mu g'(z) \tag{5}$$

where $\mu$ is the viscosity. It is easy to verify that, with these relations, the Stokes equation $\nabla p = \mu\nabla^2\mathbf{u}$ is satisfied in the fluid.

As shown by Richardson [9], velocity and stress boundary conditions on the free surface $\Gamma$ take the form

$$f'(z) + \bar{z}g'(z) - \overline{g(z)} = u_0(z)\overline{\left(\frac{dz}{ds}\right)}, \tag{6}$$

$$f'(z) + \bar{z}g'(z) + \overline{g(z)} = -i\frac{\gamma}{2\mu}\overline{\left(\frac{dz}{ds}\right)}, \tag{7}$$

where $s$ is the arclength on $\Gamma$ measured from the point of symmetry $B$ (Fig. 1(a)), $\gamma$ the surface tension coefficient, and $u_0(z)$ the (real) tangential velocity on the free surface. The manipulation of (6) and (7) yields the following equations (for $z \in \Gamma$);

$$f(z) + \bar{z}g(z) = 0, \tag{8}$$

$$\mathrm{Im}\left[\overline{\left(\frac{dz}{ds}\right)}g(z)\right] = \frac{\gamma}{4\mu}. \tag{9}$$

The conditions $u, v \to 0$ at infinity ($|z| \to \infty$) are satisfied provided

$$f(z) \to cz, \quad g(z) \to \bar{c}, \quad \text{as } |z| \to \infty, \tag{10}$$

where $c$ is an arbitrary constant. We can find from (6) and (7) that the choice $c = -i\gamma/4\mu$ is appropriate. Eqs. (2), (3) and (6)–(10) constitute the essential boundary conditions that $f(z)$ and $g(z)$ must satisfy. The symmetry conditions $\Psi = 0$, $\Omega = 0$ on $x = 0$ clearly imply that

$$\mathrm{Im}\big\{f(iy)\big\} = 0, \qquad \mathrm{Re}\big\{g(iy)\big\} = 0. \tag{11}$$

Since this problem may be considered as a generalization of previous problems, [3,7] we follow the similar procedure as in Refs. [3] and [7].

### 2.2. Conformal mapping and solution procedure

Let $z = w_1(\zeta)$, yet unknown, be the conformal mapping that maps the fluid domain $\mathcal{D}$ onto the unit disk $\mathcal{D}'$: $|\zeta| \leqslant 1$. The points $A, B, C, D, E$ of Fig. 1(a) map to the points $A', B', C', D', E'$ of Fig. 1(b), respectively. Since the mapping places the (imaged) source and sink at $\zeta = 0$, $-i\lambda$ $(0 < \lambda < 1)$, respectively, the mapping function $w_1(\zeta)$ must satisfy,

$$w_1(0) = -i, \tag{12}$$

$$w_1(-i\lambda) = -i(1 + h). \tag{13}$$

If we set,

$$F(\zeta) \equiv f\big\{w_1(\zeta)\big\} = f(z), \tag{14}$$

$$G(\zeta) \equiv g\big\{w_1(\zeta)\big\} = g(z), \tag{15}$$

$$U(\zeta) \equiv u_0\big\{w_1(\zeta)\big\} = u_0(z), \tag{16}$$

then,

$$f'(z) = \frac{F'(\zeta)}{w_1'(\zeta)}, \qquad g'(z) = \frac{G'(\zeta)}{w_1'(\zeta)}. \tag{17}$$

By the conformal mapping properties, it follows that, for $z \in \Gamma$, i.e., $|\zeta| = 1$,

$$\frac{dz}{ds} = -i\zeta\frac{w_1'(\zeta)}{|w_1'(\zeta)|}, \tag{18}$$

$$\overline{\left(\frac{dz}{ds}\right)} = \frac{i}{\zeta}\frac{\overline{w_1'(\zeta)}}{|w_1'(\zeta)|}. \tag{19}$$

Hence, boundary conditions (6) and (7) become, on $|\zeta| = 1$,

$$\frac{F'(\zeta)}{w_1'(\zeta)} + \overline{w_1(\zeta)}\frac{G'(\zeta)}{w_1'(\zeta)} - \overline{G(\zeta)} = U(\zeta)\frac{i}{\zeta}\frac{\overline{w_1'(\zeta)}}{|w_1'(\zeta)|}, \tag{20}$$