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The linear stability of double-diffusive miscible rectilinear displacements in a Hele-Shaw cell

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ABSTRACT

We investigate the viscous instability of a miscible displacement process in a rectilinear geometry, when the viscosity contrast is controlled by two quantities which diffuse at different rates. The analysis is applicable to displacement in a porous medium with two dissolved species, or to displacement in a Hele-Shaw cell with two dissolved species or with one dissolved species and a thermal contrast. We carry out asymptotic analyses of the linear stability behaviour in two regimes: that of small wavenumbers at intermediate times, and that of large times.

An interesting feature of the large-time results is the existence of regimes in which the favoured wavenumber scales with $t^{-1/4}$, as opposed to the $t^{-3/8}$ scaling found in other regimes including that of single-species fingering. We also show that the region of parameter space in which the displacement is unstable grows with time, and that although overdamped growing perturbations are possible, these are never the fastest-growing perturbations so are unlikely to be observed. We also interpret our results physically in terms of the stabilising and destabilising mechanisms acting on an incipient finger.

1. Introduction

The instabilities which occur when a fluid of high mobility displaces a fluid of low mobility represent a classic fundamental problem in fluid mechanics with important practical applications (see, for example, the review by Homsy [1]). Such instabilities occur under a variety of conditions, of which the simplest and most paradigmatic is flow in a homogeneous porous medium or the closely analogous system of a Hele-Shaw cell [2]. This problem is relevant to many industrial processes, in which instability and the subsequent fingering of one fluid into the other are typically undesirable effects, and considerable ingenuity has been devoted to eliminating them. In particular, efforts are often made to reduce the mobility of the displacing fluid below that of the fluid that it is displacing, by modifying either its temperature or its chemical composition. In nature, too, fingering instabilities may be relevant in controlling the mixing of fluids in porous rocks, and thus may affect the geochemical processes which take place and the consequent evolution of formations such as aquifers and oil reservoirs.

The earliest studies of viscous fingering (e.g. that by Saffman and Taylor [2]) considered immiscible displacements, where the displacing and displaced fluids are separated by a sharp interface; in practice capillary forces often act on this interface and provide an important stabilising mechanism. Immiscible viscous fingering is relevant to the economically important process of enhanced oil recovery, in which oil is displaced from a porous medium by water that has been made more viscous by the addition of dissolved polymers [3,4]. In some other contexts, however, there is not a sharp interface between the displaced and displacing fluids. For example, in both freshwater aquifers and geothermal reservoirs, resident water may be displaced by injected or otherwise invading water carrying different dissolved species. (The viscosity of water can be significantly affected just by the concentration of common salts in solution. For example, seawater at a salinity of 35 and temperature of 25° is around 9% more viscous than pure water at the same temperature: see [5], §§6-182 and 14-15.) As the management of these resources becomes more important, it is likely that techniques developed in the oil industry will have to be applied, for example to extract fresh water rapidly from an aquifer or to drive incompatible invading water from a geothermal system before scaling can occur. Flows which are in some respects similar occur when warm (and thus less viscous) fluid displaces colder and more viscous fluid, for example as fresh magma invades a dyke. In all these contexts, the essential destabilising mechanisms leading to viscous fingering are the same as in immiscible fingering, but a key stabilising mechanism is the thermal or molecular diffusion of the fluid properties which control viscosity. The present study is therefore concerned with miscible displacements.

Most studies of miscible displacement have assumed that the viscosity contrast between the displaced and the displacing fluids is entirely controlled either by the presence of a single dissolved species or by the temperature of the fluid. However, in many con-

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texts — for example, in the recharge of geothermal reservoirs [6] — the resident and injected fluids differ both in temperature and in composition: heat and solutes dissolve at different rates, and so the exchange of properties between them cannot be described in terms of a single diffusing quantity. Similar double-diffusive effects may occur when injected water is made more viscous by dissolving long-chain polymers in it (see e.g. [3,4]). Large polymer molecules have a substantially lower molecular diffusivity than other dissolved species (such as salts) which also affect the relative properties of the ambient and injectate, so in this situation too the effects of double diffusivity must be considered. The known complexity of gravitationally unstable displacements in double-diffusive and reactive–diffusive systems [7–9] suggests that these effects may not be simple to identify or to analyse. The object of the present study is to provide some insight into these effects by investigating the fundamental mechanisms of double-diffusive viscous fingering.

In a previous study [10], we investigated viscous fingering on a radially spreading displacement front in a porous medium, where both thermal and solutal components contributed to the viscosity contrast. In a porous medium, thermal signals are advected more slowly than the fluid because heat must be shared between the fluid and the porous matrix: the system is therefore 'double-advective' as well as double-diffusive [11], and spatially separated thermal and compositional fronts develop. The separation between these fronts controls how strongly they can stabilise or destabilise each other. By contrast, displacement processes in a well-insulated Hele-Shaw cell are 'iso-advective' and double-diffusive, and when the viscosity contrast depends on two differently diffusing dissolved species, the transport even in a porous medium becomes iso-advective. In this study, we therefore consider only iso-advective displacements: for simplicity, we phrase our discussion in terms of a displacement in a Hele-Shaw cell with a thermal and a compositional contrast, but we bear in mind that our results can be applied more generally.

A comment should be made about the validity of the Hele-Shaw flow model and its relevance as an analogue for porous media. It is key to the model that the spatial scale of all flow features is much greater than the gap width $\hat{b}$ of the Hele-Shaw cell, and that the Peclet number defined in terms of the gap width, the displacement velocity and the solutal concentration is not large. If these criteria are not satisfied then the flow becomes fully three-dimensional [12] and the gap-averaged equations defined below are no longer valid: the three-dimensional Stokes equations must be considered instead, leading to significantly different stability results [13]. Our analysis below will generally deal with relatively large length- and time-scales so this restriction is not crucial, but it should be borne in mind throughout.

A further difference between the present study and the earlier study by Pritchard [10] is that we consider a rectilinear rather than a radial displacement process. The radial geometry is particularly convenient from a mathematical perspective, since both the radius of the front and its streamwise width grow as $t^{1/2}$: it is therefore possible to eliminate the time-dependency of the basic state by transforming to similarity variables (see e.g. [14]). In other geometries, this mathematical coincidence does not occur, and it is necessary to investigate the stability of a basic state which is evolving as the perturbation develops. (Note that this issue does not arise in some analogous problems, such as reaction–diffusion systems, where a steadily translating base state is available [8].) The standard approach is to 'freeze' time at some instant, and consider perturbations to the quasi-steady state at that instant (see e.g. [15]). However, Ben, Demekhin and Chang [16] have recently argued that this approach is not justified because — especially at early times — the perturbations grow on timescales comparable to the timescale over which the basic state changes. Ben et al. developed a spectral method which could be used to obtain asymptotic results for the stability of long-wave perturbations, and we will adapt their method here. We consider rectilinear displacements, which are the simplest non-degenerate geometry available, and we believe that the insight obtained here may be more widely applicable.

This study is structured as follows. In Section 2 we formulate the governing equations. In Section 3 we carry out a linear stability analysis; we then develop asymptotic results for the growth rate of perturbations in two limits: that of long waves at intermediate times (Section 4) and that of long times (Section 5). Finally, in Section 6 we summarise our findings and their implications. Before proceeding, however, it is helpful to develop some simple hypotheses about the stability of the front, which will give us a baseline against which to compare our analysis.

1.1. Some heuristic criteria for instability

When double-diffusive viscous fingering occurs during a radial displacement, there are at least some cases in which a stabilising thermal gradient can stabilise an unstable concentration gradient [10]. It is not clear that this will be true for fingering of a rectilinear flow: the wavenumbers m are not restricted in this case to be integers, so in principle, instability at any $m > 0$ will be enough to trigger fingering. (It is generally found in stability analyses of viscous fingering that perturbations with very low wavenumbers are marginally unstable: this will be discussed further when we consider the spectral decomposition of perturbations in the streamwise direction.)

The most interesting situations occur when the temperature and compositional differences across the front contribute in different senses to the overall viscosity contrast, so one tends to stabilise and one to destabilise the front. The question is then whether the stabilising component can ever completely stabilise the front. Physical and heuristic reasoning supplies three arguments which predict substantially different results.

(i) If solute and temperature diffuse at different rates, the fates of the two fields may become decoupled. This should mean that if either component promotes fingering then fingering will occur.
(ii) For the single-species problem, in the long-wave limit the growth rates are the same as in the immiscible case [16,1]. This suggests that for the two-species problem it should be the aggregate viscosity contrast which controls at least the marginal instability of the front.
(iii) It is also intuitively plausible that instability will occur if any part of the front is potentially unstable. In other words, if the global viscosity contrast across the front is stable but the viscosity profile is not monotonic, there will be a region within the front where less viscous fluid is displacing more viscous fluid, and instability may be expected. This agrees with some previous results [17,18] for single-species fingering where the viscosity–concentration relation is non-monotonic.

We will return later to these arguments, and to the analogy with single-species fingering, in the light of our stability analysis.

2. Governing equations

We will present governing equations for flow in a Hele-Shaw cell with a temperature and a compositional contrast between the injected and ambient fluids: these equations are identical to those for the transport of two differently diffusing solutes in a porous medium, to within the limitations of the model as noted above.