European Journal of Mechanics B/Fluids 29 (2010) 208–216

Contents lists available at ScienceDirect

European Journal of Mechanics B/Fluids

journal homepage: www.elsevier.com/locate/ejmflu

Dynamics of vortex rings and spray-induced vortex ring-like structures

F. Kaplanski [a,*], S.S. Sazhin [b], S. Begg [b], Y. Fukumoto [c], M. Heikal [b]

[a] Laboratory of Multiphase Media Physics, Tallinn University of Technology, Tallinn, 12618, Estonia

[b] Sir Harry Ricardo Laboratories, Centre for Automotive Engineering, School of Environment and Technology, Faculty of Science and Engineering, University of Brighton, Brighton, BN2 4GJ, UK

[c] Graduate School of Mathematics and Mathematical Research Centre for Industrial Technology, Kyushu University 33, Fukuoka 812-8581, Japan

ARTICLE INFO

Article history:
Received 7 August 2009
Received in revised form
2 December 2009
Accepted 26 January 2010
Available online 6 February 2010

Keywords:
Vortex rings
Sprays
Gasoline engines
Vorticity

ABSTRACT

Analytical formulae, predicted by recently developed vortex ring models, in the limit of small Reynolds numbers (*Re*), are compared with numerical solutions of the underlying equation for vorticity and experimental data. Particular attention is focused on the recently developed generalised vortex ring model in which the time evolution of the thickness of the vortex ring core L is approximated as at^b, where a and b are constants ($1/4 \leq b \leq 1/2$). This model incorporates both the laminar model for $b = 1/2$ and the fully turbulent model for $b = 1/4$. A new solution for the normalised vorticity distribution is found in the form $\omega_0 + Re\,\omega_1$, where ω_0 is the value of normalised vorticity predicted by the classical Phillips solution. This solution shows the correct trends in the redistribution of vorticity due to the Reynolds number effect, and it predicts the increase in the volume of fluid carried inside the vortex ring. It is emphasised that although the structures of vortex rings predicted by analytical formulae, based on the linear approximation, and numerical calculations for arbitrary *Re* are visibly different for realistic Reynolds numbers, the values of integral characteristics, such as vortex ring translational velocity and energy, predicted by both approaches, turn out to be remarkably close. The values of velocities in the region of maximal vorticity, predicted by the generalised vortex ring model, are compared with the results of experimental studies of vortex ring-like structures in gasoline engine-like conditions with a high-pressure (100 bar) injector. The data analysis is focused on the direct measurements of droplet axial velocities in the regions of maximal vorticity. Most of the values of these velocities lie between the theoretically predicted values corresponding to the later stage of vortex ring development between $b = 1/4$ (fully developed turbulence) and 1/2 (laminar case).

1. Introduction

Vortex rings have been extensively studied theoretically and experimentally (e.g. [1,2]). Particular attention has been focused on their translational velocities and energies. Two approaches were used in theoretical studies. In the first approach, the relation between velocity and vorticity was used to obtain the formulae for thin-cored rings: $\epsilon = L/R_0 \ll 1$, where R_0 is the initial vortex ring radius and L is the core radius [3,4]. A more general approach valid for arbitrary ϵ, developed in [5] (see also [6]), is based on the Helmholtz–Lamb formula for the ring's translational velocity U in the form

$$U = \frac{\pi}{2M}\int_0^\infty \int_{-\infty}^\infty \left(\Psi - 6x\frac{\partial \Psi}{\partial r}\right)\zeta\,\mathrm{d}x\mathrm{d}r, \tag{1.1}$$

where ζ and Ψ are the vorticity and stream function, respectively, and $M = I/\rho$ is the momentum of vorticity per unit density. Using this formula, Saffman [7] (see also [8]) derived an explicit expression for the translational velocity of a thin-cored viscous vortex ring in the form

$$U_s = \frac{\Gamma_0}{4\pi R_0}\left[\ln\left(\frac{4R_0}{\sqrt{\nu t}}\right) - 0.558 + O\left(\frac{\nu t}{{R_0}^2}\ln\left(\frac{\nu t}{{R_0}^2}\right)\right)\right], \tag{1.2}$$

where Γ_0 is the initial circulation of the ring, t is time and ν is the kinematic viscosity. The vorticity distribution inside this ring corresponds to the Lamb–Oseen vortex filament [6]. This asymptotic formula is valid for the description of the initial stage of viscous vortex ring development when $\nu t \ll {R_0}^2$. The final stage of viscous vortex ring decay ($\nu t \gg {R_0}^2$) can be described based on the Phillips self-similar solution for the vorticity (ζ_f) and stream function (Ψ_f) distributions [9]:

$$\zeta_f = \frac{Mr}{16\pi^{3/2}\,(\nu t)^{5/2}}\exp\left(-\frac{s_*^2}{2}\right), \tag{1.3}$$

* Corresponding author. Tel.: +372 55621913; fax: +372 6703601.
E-mail address: Feliks.Kaplanski@ttu.ee (F. Kaplanski).

0997-7546/$ – see front matter
doi:10.1016/j.euromechflu.2010.01.002

$$\Psi_f = \frac{M}{4\pi}\left(\operatorname{erf}\left(\frac{s_*}{\sqrt{2}}\right) - s_*\sqrt{\frac{2}{\pi}}\exp\left(\frac{-s_*{}^2}{2}\right)\right)\frac{r^2}{\left(r^2+x^2\right)^{3/2}}, \quad (1.4)$$

where

$$s_* = \sqrt{\frac{r^2+x^2}{2\nu t}};$$

x, r are cylindrical coordinates for the axisymmetric vortex ring. The derivation of the translational velocity in this case is not straightforward. Since (1.1) was derived based on the full Navier–Stokes equation, the substitution of (1.3) and (1.4) into (1.1) leads to inconsistency. Attempts to account for the second-order effects of the nonlinear convective terms of the vorticity equation were made by Kambe and Oshima [10]. However, their results are not uniformly valid. Rott and Cantwell [11,12] studied this case, taking into account the flow dynamics in the potential flow region surrounding the vortical region. They showed that the asymptotic translational velocity of the ring can be predicted by the following formula:

$$U_f = \frac{7M}{15\,(8\pi\nu t)^{3/2}} = 0.0037038\frac{I/\rho}{(\nu t)^{3/2}}. \quad (1.5)$$

Another approach to this problem was developed in [13–16]. These authors obtained a first-order solution of the Navier–Stokes equation with the origin in the centre of the vortex centroid, valid in the limit of small Reynolds numbers Re defined as

$$Re = \zeta_0 L^2/\nu,$$

where $\zeta_0 = At^\lambda$ is the vorticity scale; constant A is to be specified from the conservation of M in the next section; in the same section possible choices of λ will be discussed.

The translational velocity of the viscous vortex ring was derived in the form [15]

$$U = \frac{M\theta\sqrt{\pi}}{4\pi^2R_0^3}\left\{3\exp\left(-\frac{\theta^2}{2}\right)\mathrm{I}_1\left(\frac{\theta^2}{2}\right) + \frac{\theta^2}{12}{}_2\mathrm{F}_2\left(\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},3\right\},-\theta^2\right) - \frac{3\theta^2}{5}{}_2\mathrm{F}_2\left(\left\{\frac{3}{2},\frac{5}{2}\right\},\left\{2,\frac{7}{2}\right\},-\theta^2\right)\right\}, \quad (1.6)$$

where $\theta = R_0/L = \epsilon^{-1}$, I_1 is the first-order Bessel function and ${}_2\mathrm{F}_2$ is the generalised hypergeometric function [17]. Similarly, the kinetic energy and circulation were derived in the form [16]

$$E = \frac{M^2\theta\sqrt{\pi}}{2\pi^2R_0^3}\left\{\frac{1}{12}{}_2\mathrm{F}_2\left(\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},3\right\},-\theta^2\right)\right\}, \quad (1.7)$$

$$\Gamma = \frac{M}{\pi R_0^2}\left\{1-\exp\left(-\frac{\theta^2}{2}\right)\right\}. \quad (1.8)$$

Note that, apart from the definition of Re given above, at least two other definitions of this number have been used in the literature:

$$Re_u = U_p D/\nu, \quad (1.9)$$

based on the ejection velocity U_p and orifice diameter D, and

$$Re_{\Gamma_0} = \Gamma_0/\nu, \quad (1.10)$$

where Γ_0 is the initial circulation carried by the ring.

The closed-form representations (1.6)–(1.8) enable us to analyse the asymptotic behaviour of these parameters. In the limit of small θ, these equations reduce to

$$U_f = \frac{M\theta^3}{4\pi^2R_0^3}\sqrt{\pi}\left(\frac{7}{30}-\frac{11\theta^2}{140}\right)+O\left(\theta^4\right), \quad (1.11)$$

$$E_f = \frac{M^2\theta^3}{2\pi^2R_0^3}\sqrt{\pi}\left(\frac{1}{12}-\frac{\theta^2}{40}\right)+O\left(\theta^4\right), \quad (1.12)$$

$$\Gamma_f = \frac{M\theta^2}{2\pi R_0^2}. \quad (1.13)$$

In the limit of large θ, they are reduced to

$$U_s = \frac{M\sqrt{\pi}}{4\pi^2R_0^3}\left(\frac{2\log(\theta)+3-\gamma-2\varphi(3/2)}{2}\right)+O\left(\frac{1}{\theta^4}\right), \quad (1.14)$$

$$E_s = \frac{M^2\sqrt{\pi}}{2\pi^2R_0^3}\left(\log(\theta)-\gamma/2-\varphi(3/2)\right)+O\left(\frac{1}{\theta^4}\right), \quad (1.15)$$

$$\Gamma_s = \frac{M}{\pi R_0^2}, \quad (1.16)$$

where $\gamma \approx 0.57721566$ is the Euler constant and φ is the digamma function, defined as

$$\varphi = \frac{\mathrm{d}\log\Gamma(x)}{\mathrm{d}x},$$

where $\Gamma(x)$ is the Gamma function.

Stanaway et al. [18] performed direct numerical simulation of the Navier–Stokes equation for an axisymmetric vortex ring at small and moderate Reynolds numbers. Fukumoto and Kaplanski showed that (1.6) compares fairly well with their result at a small Reynolds number [19]. The large-Reynolds-number asymptotics was discussed in [20,21].

An alternative approach for estimating the temporal evolution of the vortex ring translational velocity was suggested by Saffman [7], using simple dimensional analysis. He derived the following equation:

$$U = \frac{M}{k}\left(R_0{}^2 + k'\nu t\right)^{-3/2}, \quad (1.17)$$

where k and k' are adjustable constants.

To obtain these constants, Weigand and Gharib [22] compared their experimental results for $830 < Re_{\Gamma_0} < 1650$ with those predicted by Eq. (1.17). This comparison led them to the following values: $k = 14.4$ and $k' = 7.8$. Later, $k = 10.15$ and $k' = 8.909$ were obtained theoretically by Fukumoto and Kaplanski [19]. In contrast to the aforementioned laminar vortex ring models, the theory of turbulent vortex rings is far less developed. To the best of the authors' knowledge, the first attempt to investigate turbulent vortex ring flow structures was made by Lugovtsov [23,24] who based his analysis on the introduction of the time-dependent, turbulent (eddy) viscosity (see [25,26]):

$$\nu_* \propto L\mathrm{d}L/\mathrm{d}t. \quad (1.18)$$

Eq. (1.18) follows from a simple dimensional analysis [27], remembering that L has the dimension of length, while $\mathrm{d}L/\mathrm{d}t$ has the dimension of velocity.

Using Eq. (1.18), Lugovtsov [23,24] developed a turbulent vortex ring model with turbulent viscosity ν_*.

Eqs. (1.6)–(1.8) were originally derived for $L = \sqrt{2\nu t}$ (laminar vortex ring). Later, in [28] it was shown that they remain valid in a more general case when $L = at^b$, where a and b are constants ($1/4 \le b \le 1/2$). The model based on this presentation of L was called the generalised vortex ring model. For $a = \sqrt{2\nu}$, $b = 1/2$ and for large times (small θ), the leading-order term of (1.6) is identical with the one predicted by Eq. (1.5). For small times, $\nu t \ll R_0^2$, the vorticity is concentrated on a circle of radius R_0 and tends to a Gaussian form [15]. The leading term of (1.14) coincides with Saffman's formula, (1.2) [19].

For $b = 1/4$ in the generalised vortex ring model, the leading term in (1.11) corresponds to