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Large-amplitude steady rotational water waves

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Abstract

Two-dimensional, finite-depth periodic water waves with general vorticity and large amplitude are computed. The mathematical formulation and numerical method that allow us to compute a continuum of such waves with arbitrary vorticity are described. The problems of whether extreme waves exist, where their stagnation points occur, and what qualitative features such waves possess are addressed here with particular emphasis on constant vorticity.

Keywords: Vorticity; Rotational water waves; Extreme waves; Stagnation

1. Introduction

Evidence from experimental studies shows that qualitative features of water waves can be greatly affected by the presence of vorticity. However, numerical studies on rotational water waves have mostly been restricted to the simplified setting of constant vorticity. These studies have consistently shown that features such as the shape of the wave, the amplitude of the crest, and the presence of eddies differ from those found in the irrotational setting. There may be “extreme” waves that contain points of stagnation; they indicate the possibility of eddies or of overhanging crests. The irrotational case is the only one for which there is a complete picture of the “extreme” wave, that is, a wave with a stagnation point. This is the Stokes’ wave, for which the only point of stagnation is the crest with an angle measuring 120°. Theoretical work concerning extreme rotational waves is quite sparse.

In the present work, we compute families of periodic water waves with general vorticity and large amplitude. The vorticity does not have to be constant; it is completely arbitrary. Our calculations make no shallowness or small-amplitude approximation. We do assume that the water is incompressible and inviscid without surface tension, lies over a flat bottom, and is acted upon by gravity g. We assume that the waves are two-dimensional, periodic and of permanent form. Then the only remaining free parameters are the period $2L$, the Bernoulli constant Q, the relative mass flux p_0, the speed c of the wave, and of course the vorticity function $\gamma(\cdot)$. The average depth d is determined implicitly in terms of the other parameters. We treat Q as a bifurcation parameter, thereby generating a one-parameter family of waves, for each choice of the other parameters.

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Our principal aim is to compare waves with different vorticities, especially with regard to the possibility of extreme waves with stagnation points. We are interested to see what these extreme waves look like and where the stagnation points are located. Key conclusions that are reached from our computations in the case of constant vorticity are as follows:

(1) There can be stagnation, not only at the crest, but also at the point on the bottom directly below the crest.
(2) These are the only two possibilities for a first stagnation point.
(3) The shapes of the streamlines of the extreme waves depend on the vorticity.
(4) Furthermore, in case the vorticity is variable, there can also be stagnation points inside the fluid.

There is of course a huge literature concerning irrotational waves, especially in various small-amplitude and shallow-water approximations. In case the vorticity ω is a *constant*, one can introduce a pseudo-stream function that is a harmonic function, in analogy with the irrotational case. Thus the Euler equations can be converted via the Cauchy integral theorem to a boundary-integral formulation. This formulation is relatively easy to solve numerically. The first to use this approach were Simmen and Saffman in [1] who considered periodic waves with infinite depth and Da Silva and Peregrine in [2] who treated periodic waves with finite depth. Da Silva and Peregrine showed that eddies can form at the finite bottom directly below the crest if the vorticity is positive and large enough. In case the vorticity is negative, an extreme wave can form with stagnation at the crest but it does not have the same shape as in the irrotational case. Their key paper clearly shows that vorticity can have a profound effect on the shape of the wave.

Using the same method, Vanden-Broeck in [3,4] and Okamoto and Shoji in [5] computed various families of solitary waves with finite depth and constant vorticity, obtaining results consistent with those of Da Silva and Peregrine. Miroshnikov in [6] also considered solitary waves using basically the same formulation. However, in his work the water has to be shallow and the numerical method is based on several approximations including an expansion in powers of the depth. As Da Silva and Peregrine did, Miroshnikov found eddies forming at the bottom if the vorticity is positive, but, in direct contradiction to the results of both Da Silva and Peregrine and Vanden-Broeck, he also found eddies forming near the crest.

A few papers have dealt with the case of *variable vorticity*. Dalrymple in [7] used a method based on the Dubreil-Jacotin transformation (DJ, see below), which permits the treatment of an arbitrary vorticity distribution. In fact, Dubreil-Jacotin [8] had been the first to provide any theoretical analysis of waves with general vorticities. Dalrymple specified all the physical constants and therefore he computed only particular examples in each run. He computed just two particular examples in detail, one with constant vorticity and one with a vorticity satisfying a power law.

Thomas in [9] also used the Dubreil-Jacotin method similarly to Dalrymple, computing individual waves, but included a background current and performed the computation using truncated Fourier modes. Solitary waves in the presence of a background current had been considered earlier by [10]. Comparing his numerical results with experimental data, Thomas' key conclusion was that the vorticity has a major influence on the nature of the wave.

Swan, Cummins and James in [11] undertook an experimental and numerical study of time-dependent waves propagating on a strongly sheared current with a nonconstant vorticity distribution. They found, experimentally, that a negative vorticity distribution can have several main effects on the gross appearance of a wave. It may produce increased wave amplitude. There may be greater crest-trough asymmetry (with a broader trough and a sharper crest). The wave may be steeper than in the irrotational case and thus the local acceleration of the water particles can be greater. Even if the vorticity is confined to the upper layers of the water, the flow is modified over the entire water depth. (We remark that what SCJ call "negatively sheared" is what we call "positive vorticity", and vice-versa.)

In this paper we follow the bifurcating curve using the parameter Q, after having fixed the relative mass flux p_0 and the period $2L$. We do not use truncated Fourier modes, nor shallow water or small amplitude approximations. Instead, we solve the fully elliptic system that results from the DJ transformation along the bifurcating curve, using standard finite differencing and nonlinear solvers, as well as the efficient numerical continuation library TRILINOS.

In Section 2 we present the theoretical background of the problem, including the DJ transformation and a summary of known mathematical results, especially those that concern the stagnation points. In Section 3 we describe our numerical method, including discussions of the numerical bifurcation and continuation procedures. In Section 4 we present the results of our simulations in the case of constant vorticity. There are three natural regimes, depending on whether the vorticity is sub-irrotational, supercritical, or in between. We conclude Section 4 with one instance of a