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Transport of reactive solutes in unsteady annular flow subject to wall reactions

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ABSTRACT

The present paper concerns with the dispersion process in steady and oscillatory flows through an annular pipe in presence of reversible and irreversible reactions at the wall. Method of homogenization, a multiple-scale method of averaging, is adopted for deriving the effective transport equations. The main objective is to look into the effect of aspect ratio of the annular pipe on the dispersion coefficient due to the combined effect of axial convection and radial diffusion in steady and oscillatory flows along the annulus, subject to the kinetic reversible phase exchange and irreversible absorption at the outer wall. Results demonstrate that upto a certain critical value of aspect ratio, dispersion coefficient increases with increase of aspect ratio when the wall is retentive, though the wall inertness may lead to decrease of dispersion coefficient with increase of aspect ratio. The results would be useful to the medical practitioners working in the domain of catheterized artery.

1. Introduction

The study of longitudinal dispersion of tracers through an annular straight tube is of considerable interest due to its applications in the field of chemical, environmental and biomedical engineering. The first fundamental study on dispersion was that of Taylor [1] who showed that the dominant mechanism, whereby a scalar contaminant cloud (or solute) is spread in the steady laminar flow within straight tube, is dispersion – the interaction between non-uniform velocity and diffusion across the flow. Aris [2] extended Taylor's theory to include longitudinal diffusion and developed an approach 'method of moments' to analyze the asymptotic behaviour of second order moment about the mean.

An exact solution of the diffusion equation was obtained by Chatwin [3] to study the dispersion in oscillatory flow. Watson [4] used the concept proposed by Chatwin to study the passive contaminant dispersion in an oscillatory pressure-driven flow. Jimenez and Sullivan [5] used a probabilistic model to study the stream wise dispersion in unsteady laminar flow. Pedley and Kamm [6] studied the axial mass transport in an annular region in presence of an oscillatory flow field.

There exist a large number of studies on Taylor dispersion under the sole influence of irreversible reaction. Some of them considered channel flow (Mondal and Mazumder [7]) or tube flow (Jiang and Grotberg [8]) while in few cases annular tube flows (Sarkar and Jayaraman [9], Mazumder and Mondal [10]) were also considered. Dispersion coefficient affected by a reversible phase exchange has been studied by Davidson and Schroter [11]; Phillips and Kaye [12]. But very few studies includes the effect of both reversible and irreversible reactions. Though Purnama [13], Revelli and Ridolfi [14,15] and Ng [16,17] carried out this effect for flow through a tube or open-channel, but to the best of our knowledge no attempt has been made to examine dispersion phenomena through an annular tube considering both reversible and irreversible wall reactions.

Effect of aspect ratio on dispersion was studied by Mondal and Mazumder [18], Mazumder and Mondal [10], Sarkar and Jayaraman [9,19] and others. These studies reveal that aspect ratio of annular pipe has important contribution in dispersion process. But in all these cases, only the effect of irreversible reaction at the boundary is considered, though the model demands to have application in catheterized artery where the reversible reaction also plays important role.

The main objective of the present paper is to examine the influence of annularity on the transport process under the combined effects of reversible and irreversible wall reactions, when the flow is driven by a pressure gradient comprising of steady and periodic components. The inner wall of the outer tube is lined with a very thin layer made up of a retentive and reactive materials. The substance that undergoes Taylor dispersion in the fluid is subject to reversible phase exchange and irreversible absorption. These boundary reactions are the most important processes controlling the dispersion of solute in catheterized artery. A relatively fast rate of phase exchange and a relatively slow rate of absorption

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were considered. Results are shown how the spreading of tracers is influenced by the aspect ratio, phase exchange parameter and absorption.

The insertion of catheter into an artery leads to a formation of an annular region between the catheter and the arterial wall (McDonald [20]). The fact that the lung and blood vessels have conductive walls (where phase exchange between the arterial wall and the flowing fluid (blood) and reaction with a reactant secreted by the wall tissue may take place) and the catheter, of course, acts as an impermeable boundary is reflected in our boundary conditions. The model will help us in understanding the indicator technique and other mechanisms in the branchial region. The insertion of a pipe with smaller diameter at the centerline of an artery brings the asymmetry to the flow, and the increase of aspect ratio leads to the existence of symmetry of the annular flow.

2. Velocity distribution

We consider a fully developed, axi-symmetric laminar flow of a homogeneous, incompressible viscous fluid through an annular pipe having inner radius b and outer radius a (i.e., $a > b$). We have used a cylindrical coordinate system in which the radial and axial co-ordinates are r and x respectively. The flow is assumed to be unidirectional and so the velocity has only axial component $u(r,t)$ which satisfies the Navier–Stokes equation as:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

where $\partial p/\partial x$ is the axial pressure gradient, ρ is the fluid density and ν is the kinematic viscosity.

The horizontal pressure gradient, which drives the flow, consists of steady and harmonically fluctuating components,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = P\left[1 + \psi\,\mathrm{Re}\left(e^{i\omega t}\right)\right]$$

where $P > 0$ is the steady part of the pressure gradient, ψ is a factor such that $P\psi$ is the amplitude of the oscillatory part of the pressure gradient.

The no-slip conditions on the surface of the inner and outer walls of the annular pipe i.e. $u(b,t) = 0$ and $u(a,t) = 0$, will produce the following velocity profile,

$$u(r,t) = u_s(r) + \mathrm{Re}\left[u_w(r)e^{i\omega t}\right] \tag{1}$$

where the steady component $(u_s(r))$ is given by,

$$u_s(r) = K\langle u_s\rangle\left[1 - \left(\frac{r}{a}\right)^2 - \frac{1-(b/a)^2}{\log(b/a)}\log\left(\frac{r}{a}\right)\right] \tag{2}$$

where $\langle u_s\rangle$ is the velocity averaged over the annular section given by

$$\langle u_s\rangle = \frac{Pa^2}{4\nu K} \quad \text{and} \quad K = \frac{2}{1 + (b/a)^2 + (1-(b/a)^2)/\log(b/a)}$$

and the unsteady component $(u_w(r))$ of the velocity is

$$u_w(r) = c_1 K_0\left(\frac{1-i}{\delta}r\right) + c_2 I_0\left(\frac{1-i}{\delta}r\right) - \frac{iP\psi}{\omega} \tag{3}$$

where

$$c_1 = \frac{iP\psi}{\omega}\left[\frac{I_0(\frac{1-i}{\delta}b) - I_0(\frac{1-i}{\delta}a)}{K_0(\frac{1-i}{\delta}a)I_0(\frac{1-i}{\delta}b) - K_0(\frac{1-i}{\delta}b)I_0(\frac{1-i}{\delta}a)}\right],$$

$$c_2 = \frac{iP\psi}{\omega}\left[\frac{K_0(\frac{1-i}{\delta}b) - K_0(\frac{1-i}{\delta}a)}{I_0(\frac{1-i}{\delta}a)K_0(\frac{1-i}{\delta}b) - I_0(\frac{1-i}{\delta}b)K_0(\frac{1-i}{\delta}a)}\right]$$

and

$$\delta = \sqrt{\frac{2\nu}{\omega}}.$$

Here I_0 and K_0 are respectively the modified Bessel function of first kind and second kind of order zero with imaginary arguments. The function I_0 and K_0 can be expressed as $I_0(ri^{1/2}) = \mathrm{ber}(r) + i\,\mathrm{bei}(r)$ and $K_0(ri^{1/2}) = \mathrm{ker}(r) + i\,\mathrm{kei}(r)$ where ber, bei, ker and kei are Kelvins functions of order zero.

For flow through a tube (i.e., when $b = 0$), the steady and unsteady components of the velocity are given by

$$u_s(r) = 2\langle u_s\rangle\left[1 - \left(\frac{r}{a}\right)^2\right] \quad \text{with } \langle u_s\rangle = \frac{Pa^2}{8\nu}$$

and

$$u_w(r) = -\frac{iP\psi}{\omega}\left[1 - \frac{J_0(\frac{1-i}{\delta}r)}{J_0(\frac{1-i}{\delta}a)}\right].$$

3. Governing equation and boundary conditions

Let us consider the transport of a chemical species through the annular gap of a tube. If the species is completely miscible with the fluid and $C(x,r,t)$ is the concentration (mass of species dissolved per bulk volume of the fluid) of the mobile phase, then C satisfies the mass transport equation as:

$$\frac{\partial C}{\partial t} + u(r,t)\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} + \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right), \quad b < r < a \tag{4}$$

along with the boundary conditions,

$$\frac{\partial C}{\partial r} = 0, \quad r = b, \tag{5}$$

$$-D\frac{\partial C}{\partial r} - \Gamma C = \frac{\partial C_s}{\partial t} = k(\alpha C - C_s), \quad r = a, \tag{6}$$

where D is the molecular diffusion coefficient assumed to be constant and $C_s(x,t)$ is the concentration (mass of species retained per unit surface area of the wall) of the immobile phase. Here Γ, k and α are the irreversible absorption rate, the reversible reaction rate and the partition coefficient respectively.

The boundary condition (5) states that there is no net transport of mass through the inner wall of the annular pipe. Last equality of the boundary condition (6) states that the rate of accumulation of the immobile phase in the outer wall is linearly proportional to the departure from local equilibrium between concentrations of the two phases on the outer wall of the annulus, and the first equality simply describes the irreversible reaction occurring at the surface of the outer wall.

4. Assumptions

The following assumptions are made for carrying out the perturbation analysis,

1. The length scale for the longitudinal spreading of the chemical cloud is much greater than the annular gap. It is meant that $x = \mathrm{O}(L)$ and $r = \mathrm{O}(a-b)$, where L is a characteristic longitudinal distance for the chemical transport. The ratio

$$\epsilon = (a-b)/L \ll 1 \tag{7}$$

is small enough to use as ordering parameter.
2. The oscillation period of the flow is so short that within this period there are no appreciable transport effects along the tube, though the effect of radial diffusion is not negligible. But the annular gap of the tube is so fine that diffusion across the entire annular section may be accomplished within this short time scale.
3. The two reactions are of different orders. The reversible phase exchange is much faster than the irreversible reaction. This ensures that local equilibrium can be largely achieved over a finite number of oscillations.