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A numerical method for the computation of bifurcation points in fluid mechanics

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Abstract

Two original algorithms are proposed for the computation of bifurcation points in fluid mechanics. These algorithms consist of finding the zero values of a specific indicator. To compute this indicator a perturbation method is used which leads to an analytical expression of this indicator. Two kinds of instability are considered: stationary and Hopf bifurcations. To prove the efficiency and advantages of such numerical methods several numerical tests are discussed.

Keywords: Bifurcation indicators; Hopf bifurcation; Asymptotic numerical method

1. Introduction

In this paper, we are interested in numerical investigations of two types of classical instability: stationary and Hopf bifurcations. A stationary bifurcation corresponds to a transition from a steady flow (usually symmetric) to another flow (in most cases, with nonsymmetric solutions), whereas a Hopf bifurcation indicates the appearance of a time-periodic solution from a steady branch. The conditions necessary for encountering a Hopf bifurcation are as follows: [1,2] a stationary solution exists for which two of the eigenvalues of the Jacobian matrix cross the imaginary axis. This means that if ζ_k are the eigenvalues of the Jacobian matrix, then at a Hopf bifurcation point, two eigenvalues are purely imaginary (i.e. $\zeta_1 = \overline{\zeta}_2 = \mathrm{i}\omega$, $\mathrm{Re}(\zeta_k) \neq 0$ for $k \geqslant 3$).

Common methods for computing Hopf bifurcation points precisely are generally divided into two families: the “indirect” and the “direct” method. The indirect method consists of finding the solutions of the equation $\mathrm{Re}(\zeta) = 0$ along the stationary solution branches. This method requires, for each steady state solution, the computation of the eigenvalues of the Jacobian matrix [3]. In some cases, when the number of unknowns of the system is too high, it is sometimes impossible or at least highly prohibitive to compute all eigenvalues. In such cases, only part of the spectrum is calculated [4] using, for example, an Arnoldi algorithm [5].

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The direct method consists of solving an augmented system whose solutions are Hopf bifurcation points [6,7]. This method does not need to follow the stationary solution branch, but it requires good initial values (a good initial Reynolds number and a good approximation of the initial eigenvalue) to obtain an acceptable convergence.

In this paper, we propose another method, which can also be included in the family of indirect methods using a different function (not, $\mathrm{Re}(\zeta) = 0$) to characterize the bifurcation points. This method is based on the introduction of indicators with the distinctive feature of having zero values at singular points. These indicators were first developed in structural stability mechanics. Boutyour et al. [8] have defined a stationary indicator. Bensaadi [9] has proposed theoretical work on the determination of Hopf bifurcation and Tri et al. [10] have applied this theory in structural problems. We use these indicators and adapt them for fluid instability problems.

The basic idea is to perturb the stationary solution (written U_λ) by a load vector μf, where μ is the intensity and f is a given vector force. The linearized and perturbed problem arising from this perturbation can then be written in the following form:

$$\pounds_T(U_\lambda, \omega) \cdot \Delta\mathcal{V} = \mu f \tag{1}$$

where $\Delta\mathcal{V}$ is the fluctuation in the velocity resulting from the perturbation force. $\pounds_T(U_\lambda, \omega)$ is an operator which depends on the Reynolds number (via the fundamental stationary solution U_λ) and, for the Hopf bifurcation, of the angular frequency ω. This operator depends on whether it is a stationary or a Hopf bifurcation that we are looking for. In Eq. (1) one can see that when μ is null, then (1) is equivalent to an eigenvalue problem. So the quantity μ is our indicator of bifurcation (as will be shown later): computation of the bifurcation points consists of finding the point of the fundamental solution branch where μ is equal to zero.

When stationary bifurcations are considered, the indicator only depends on the fundamental point, U_λ. This stationary solution is computed using a perturbation method combined with the finite element method (Asymptotic Numerical Method [11–14], "ANM"). Thus, U_λ is an explicit analytical function of a path following parameter "a". Considering the previous idea, the bifurcation indicator only depends on this path parameter. This property has been used by Boutyour et al. [8] and Tri et al. [15] to determine the stationary bifurcation points in structural and fluid mechanics. They used a perturbation method to compute both the indicator μ and the vector ΔV. The advantages of such a method are as follows: On one hand, in order to determine the quantities μ and ΔV the operator $\pounds_T(U_\lambda)$ is the same as that used to compute the stationary solution branches. This therefore leads to a very small increase in CPU time to evaluate the indicator as compared with the CPU time required to compute the stationary solution. On the other hand, when solving a nonlinear problem with ANM, the solutions are continuously known, as is the indicator. The indicator is then determined along the whole solution branch and not just for some points as the quantity $\mathrm{Re}(\zeta)$ with the indirect method mentioned above.

In the case of a Hopf bifurcation, the indicator depends on the path following parameter "a" and also on the angular frequency ω. So Tri et al. [10] have used a perturbation method with these two parameters (a and ω). However this method is very difficult to apply and it is almost impossible to build an automatic method for detection of bifurcation points in this way. To overcome these two drawbacks, we propose another way here, which consists of setting the Reynolds number and taking the angular frequency as the perturbation parameter. In this way we can easily use a perturbation method and also a continuation method (Cochelin [11]) to compute the indicator for each value of the angular frequency.

The first part of this paper is devoted to the determination of stationary bifurcation points on stationary branches, which are solutions of the Navier–Stokes equations. The second part proposes the determination of the first Hopf bifurcation on the stationary solution branches of the Navier–Stokes equations.

2. Bifurcation indicators for fluid mechanics

We consider a viscous incompressible fluid whose motion is governed by the stationary Navier–Stokes equations:

$$\begin{cases} -\nu u_{i,jj} + u_j u_{i,j} + \dfrac{1}{\rho} p_{,i} = 0 & \text{in } \Omega, \\ u_{i,i} = 0 & \text{in } \Omega \end{cases} \tag{2}$$