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Implicit–explicit schemes of finite element method for the non-stationary thermal convection problems with temperature-dependent coefficients ☆

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ABSTRACT

For the time-dependent thermal convection problems with temperature-dependent coefficients, the implicit–explicit scheme is presented, in which mixed finite element method is applied for the spatial approximation of the velocity, pressure and temperature while the time discretization is based on the high-order backward difference scheme. Linear terms are dealt with the implicit scheme while the nonlinear terms are treated by the semi-implicit scheme. The advantages for this scheme are unconditionally stable, decoupled computational and second order accuracy. Finally, numerical tests illustrate the theoretical results of the presented schemes, and display that the highly efficient method conserves the property of divergence free of the original problems.

1. Introduction

The Boussinesq model is one of the most useful models in fluid and geophysical fluid dynamics. This model also constitutes an important system of equations in atmospheric dynamics and a dissipative nonlinear system of equations. Since this system does not contain only the velocity and pressure fields but also the temperature field. Note that the convection is a nonlinear transport process in this model. Hence, it is difficult to achieve long time numerical simulations for this model. Thus, development of an efficient computational method for investigating this model has drawn the attentions of many researchers because of many real applications. There are so far numerous works devoted to the development of efficient numerical schemes for the conduction–convection equations with constant coefficients [1–8]. For example, the finite element method, the penalty method, the variational multiscale method, etc.

Recently, a lot of experiments have shown that the variation of viscosity with the temperature is an important factor for details of the flow, for example, the process of glass production and the mantle convection inside. In particular, the viscosity depends strongly on the temperature of the mantle flow. So the nonlinear natural convection model with temperature-dependent coefficients has drawn the attention. From the mathematical point of view, Lorca and Boldrini [14] have first used the spectral Galerkin method and proved the existence of global weak solutions in 3D and local strong solution in 2D to the initial value problem for a generalized Boussinesq model under mild assumptions on the temperature dependency of the viscosity and thermal conductivity and no-slip boundary condition. In 2009, Gunzburger et al. [16] established the well-posedness of the infinite Prandtl number model for convection with temperature-dependent viscosity under the free-slip boundary condition and zero horizontal fluxes. As far as we know, there are few researches on the fully discrete methods for the nonstationary thermal convection problems with variable coefficients, since the convection term is a nonlinear transport process in our model. Based on the energy method, Tabata and Tagami [15] in 2005 derived the error estimates of finite element methods for this model with temperature-dependent coefficients by using the first-order backward Euler method in time and the conforming finite elements in space without numerical tests. They didn't consider the high-order time discretization schemes and the influence of the Prandtl number

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and the Rayleigh number on the solution of the nonlinear natural convection model.

In general, there exist three classical schemes to deal with the time-dependent problems, which are the fully implicit, implicit–explicit, and explicit schemes. Among them, high-order schemes are of more interest because the first-order schemes are not sufficiently accurate for long time approximations. Meanwhile, the stability condition of schemes is also a key issue. Usually an explicit scheme is easy in computation, but it suffers a severely restricted time step size from stability requirement. Although the fully implicit schemes are (almost) unconditionally stable, one has to solve a system of nonlinear equations at each time step. Hence, we often use the implicit–explicit schemes to solve the nonlinear problems. Namely, we adopt an implicit scheme for the linear terms and an implicit–explicit scheme for the nonlinear terms. Moreover, the well-known high-order backward difference formula is one of this approach. Recently, Su et al. [5] proposed a second-order accuracy scheme based on Crank–Nicolson extrapolation for the two-dimensional time-dependent conduction–convection equations with constant coefficients, which is almost unconditionally stable. However the stability of this scheme is not good enough compared with other second-order implicit–explicit schemes for long time numerical simulations.

In this paper, we are concerned with the implicit–explicit decoupled finite element scheme for the nonstationary conduction–convection equations with temperature-dependent coefficients. Based on the high-order backward difference scheme, the first-order and second-order implicit–explicit fully discretized schemes are proposed respectively, in which we use an implicit scheme for the linear terms and the semi-implicit schemes for the nonlinear terms. Mixed finite element method (e.g. $P_2-P_1-P_2$and$P_1b-P_1-P_1$) is applied for the spatial approximation of the velocity, pressure and temperature. Then the numerical results illustrate the high efficiency of the proposed schemes. Note that these schemes guarantee the property of solenoidal vector field for the original problem in a way.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations, an abstract functional setting of the time-dependent thermal convection problems with temperature dependent coefficients. Mixed finite element strategy is recalled and some well-known results used throughout this paper in Section 3. Fully discrete method based on the implicit–explicit decoupled scheme is given in Section 4. In Section 5, numerical experiments are given to verify the theoretical results completely. Finally, we end with a short conclusion in Section 6.

2. Preliminaries

Let Ω be a bounded, convex and open subset of $\mathbb{R}^d(d=2,3)$ with a Lipschitz continuous boundary $\partial\Omega$. We consider the following time-dependent thermal convection problems with temperature-dependent coefficients

$$\begin{cases} Pr^{-1}\left(u_t+(u\cdot\nabla)u\right)-\nabla\cdot(\nu(\theta)D(u))+\nabla p-Ra\beta(\theta)\theta=f, & \text{in}\quad \Omega\times(0,T],\\ \nabla\cdot u=0, & \text{in}\quad \Omega\times(0,T],\\ \theta_t-\nabla\cdot(\kappa(\theta)\nabla\theta)+u\cdot\nabla\theta=g, & \text{in}\quad \Omega\times(0,T],\\ u(x,0)=u_0,\quad \theta(x,0)=\theta_0, & \text{on}\quad \Omega\times\{0\},\\ u=u_D,\quad \theta=\theta_D & \text{in}\quad \partial\Omega\times(0,T], \end{cases} \tag{1}$$

where $u\in\mathbb{R}^d$ represents the velocity vector, $p=p(x,t)$ the pressure, $\theta=\theta(x,t)$ the temperature, Pr the Prandtl number, which is the ratio of the representative value of the kinematic viscosity to the representative value of the thermal diffusivity, and Ra the Rayleigh number, which denotes the ratio of relative heating to the overall dissipation. $D(u)=\frac{1}{2}(\nabla u+\nabla u^T)$ is the symmetrized deformation tensor. $\nu,\kappa\in\mathbb{R}^+$ and $\beta\in\mathbb{R}^d$ denote the generalized viscosity, thermal conductivity, and thermal expansion coefficients depending on the temperature respectively. Such dependence could be of great importance in cases of the large temperature in some certain applications. (u_0,θ_0) and (u_D,θ_D) denote the initial value and boundary value respectively. f and g denote the external force and heat source respectively. T is the given final time and $u_t=\partial u/\partial t$, $\theta_t=\partial\theta/\partial t$. Furthermore, assume the given functions $\nu,\quad\kappa\in C(\bar\Omega\times[0,T])$ and $\beta\in C(\bar\Omega\times[0,T])$.

With the standard Sobolev spaces

$$X=(H^1(\Omega))^d,\quad X_0=(H_0^1(\Omega))^d,\quad W=H^1(\Omega),\quad W_0=H_0^1(\Omega),$$
$$M=L_0^2(\Omega)=\{q\in L^2(\Omega):\int_\Omega q\,dx=0\},$$

a weak formulation of Eq. (1) reads: find $(u,p,\theta)\in(X,M,W)$ for all $t\in(0,T]$ such that for all $(v,q,s)\in(X_0,M,W_0)$ and $(u,\theta)|_{\partial\Omega}=(u_D,\theta_D)$,

$$\begin{cases} Pr^{-1}\left((u_t,v)+b(u;u,v)\right)+B_\nu((u,p);(v,q))-a_\beta(\theta,v)=(f,v),\\ (\theta_t,s)+a_\kappa(\theta,s)+\bar b(u;\theta,s)=(g,s),\\ u(x,0)=u_0,\quad \theta(x,0)=\theta_0, \end{cases} \tag{2}$$

with

$$a_\nu(u,v)=(\nu(\theta)\nabla u,\nabla v),\quad d(v,q)=(q,\mathrm{div}v),\quad a_\kappa(\theta,s)=(\kappa(\theta)\nabla\theta,\nabla s),$$
$$b(u;v,w)=((u\cdot\nabla)v,w)+\frac{1}{2}(w(\mathrm{div}u),v)=\frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v),$$
$$\bar b(u;\theta,s)=((u\cdot\nabla)\theta,s)+\frac{1}{2}(\theta(\mathrm{div}u),s)=\frac{1}{2}((u\cdot\nabla)\theta,s)-\frac{1}{2}((u\cdot\nabla)s,\theta),$$
$$B_\nu((u,p);(v,q))=a_\nu(u,v)-d(v,p)+d(u,q),\quad a_\beta(\theta,v)=(Ra\beta(\theta)\theta,v).$$

The trilinear forms $b(\cdot;\cdot,\cdot)$ and $\bar b(\cdot;\cdot,\cdot)$ satisfy

$$\begin{aligned} &|b(u;v,w)|\le N\,\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0,\quad \forall u,v,w\in X,\\ &|\bar b(u;\theta,s)|\le \bar N\,\|\nabla u\|_0\|\nabla\theta\|_0\|\nabla s\|_0,\quad \forall(u,\theta,s)\in(X,W,W), \end{aligned} \tag{3}$$

where

$$N=\sup_{u,v,w\in X}\frac{|b(u;v,w)|}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0},\quad \overline{N}=\sup_{u\in X,\theta,s\in W}\frac{|\bar b(u;\theta,s)|}{\|\nabla u\|_0\|\nabla\theta\|_0\|\nabla s\|_0}.$$

Using the similar techniques in [14–16], if the spaces X and M satisfy the continuous inf-sup condition, the existence and uniqueness of solution (u,p,θ) for problem (2) can be obtained under some reasonable propositions on the given functions f,g,ν,κ and β, the initial boundary conditions (u_0,θ_0) and (u_D,θ_D), and the smoothness of boundary. And the maximum principle of this problem can be also obtained. Interested readers can refer to [14, 15] for details.

3. Mixed finite element method

Now we introduce the standard finite-dimensional subspaces $(X_h,M_h,W_h)\subset(X,M,W)$ which are characterized by K_h, a partitioning of Ω into triangles K with the mesh size $h\in(0,1)$, assumed