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Analysis of hyperbolic heat conduction in 1-D planar, cylindrical, and spherical geometry using the lattice Boltzmann method☆

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ABSTRACT

Analyses of hyperbolic heat conduction in an 1-D planar, cylindrical, and spherical geometry are analyzed using the lattice Boltzamnn method (LBM). Finite time lag between the imposition of temperature gradient and manifestation of heat flow causes the governing energy equation to be hyperbolic one. Temporal temperature distributions are analyzed for thermal perturbation of a boundary by suddenly raising its temperature and also by imposing a constant heat flux to it. Wave-like temperature distributions in the medium are obtained when constant temperature boundary condition is used. However, when constant heat flux boundary condition is used, temperature distribution fluctuates before it becomes stable. To check the accuracy of the LBM results, the problems are also solved using the finite difference method (FDM). LBM and FDM results compare exceedingly well. LBM has computational advantage over the FDM.

1. Introduction

Nothing is known to be instantaneous, and without a cause. A time lag Γ must exist between the cause and the effect(s). In heat transfer by conduction, as long as temperature gradient ∇T (cause) exists, heat transfer q (effect) manifests. However, according to Fourier's law of heat conduction $q = -k\nabla T$,where k is the thermal conductivity, there is no time lag between the cause and the effect, they manifest simultaneously, in other words, heat propagates with an infinite speed. This is contrary to the observations which include but are not limited to analyses of heat conduction transfer at very small spatial and temporal dimensions with and without periodic boundary conditions [1,2], in a material subjected to short-pulse laser [3,4], thin films/plates [5–7], material with inhomogeneous structure [8,9], etc. Contrary to observations [1–9], omission of time lag Γ term, in other words, infinite speed of propagation of energy in conduction heat transfer, precludes wave-like nature in theoretical analysis based on Fourier's law of heat conduction.

The aforesaid anomaly in prediction by Fourier's law of heat conduction and observed fact [1–9] has been resolved by incorporating a finite time lag Γ in the manifestation of heat flux $q$as proposed by Cattaneo [10] and Vernotee [11] some six decades back.

$$q(r, t+\Gamma) = -k\nabla T(r, t) \tag{1}$$

Eq. (1) accounts for the finite propagation speed $C$of energy transfer by conduction, and with this, the governing energy equation mathematically turns out to be hyperbolic one, and in literature, it is also known as the hyperbolic heat conduction (HHC) equation. It is to be noted when the relaxation time $\Gamma = \frac{\alpha}{C^2} = 0.0$ or the speed of propagation $C = \infty$, Eq. (1) turns out to be Fourier one, and the nature of the corresponding governing energy equation is parabolic. Following Cattaneo [10] and Vernotee [11], many heat transfer studies [1–15] covering a wide range of problems have been reported. Researchers have used different numerical methods to analyze problems.

Recently, there has been a surge in the application of the lattice Boltzmann method (LBM) to a wide range of fluid flow and heat transfer problems [12–19]. This surge is owing to its simplicity in formulation, implementation of boundary conditions, and computational advantage over other methods like the finite difference method (FDM), the finite element method, and the finite volume method [13,14]. Some researchers [15–17] have also used LBM in the analyses of HHCE in different geometry. Most of such studies pertain to thermal perturbation caused by raising temperature of one of the boundaries [17].

Response of a system depends on how one or more of its boundaries are thermally perturbed. Thermally, a boundary can be perturbed either by changing (raising or lowering) its temperature or imposed heat flux. How for the two types of boundary conditions, temporal temperature distributions vary for HHC has not been reported. Thus, the present work aims at the application of the LBM to analyze HHC in an 1-D planar, cylindrical, and spherical geometry for both types of boundary conditions. To check the accuracy of results and computational efficiency of the LBM, the same problems are also solved using the FDM.

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Nomenclature

A	Area
c_p	Specific heat
C	Speed of thermal wave
$\vec{e}_i$	Propagation velocity in the direction i in the lattice
f_i	Particle distribution function in the i direction
$f_i^{(0)}$	Equilibrium particle distribution function in the i direction
k	Thermal conductivity
n	Index for geometry: 0—planar, 1—cylindrical, and 2—spherical

Greek Symbols

α	Thermal diffusivity
Θ	Non-dimensional temperature
ζ	Scaled non-dimensional time
ξ	Scaled non-dimensional time ($=\zeta-\zeta_0$)
ρ	Density
τ	Relaxation time
Γ	Time lag

Subscripts

ref	Reference value
o	Initial value

Superscripts

*	Dimensional quantities
(n)	Nth order term in the Chapman–Enskog expansion

2. Formulation

For the 1-D geometry (planar, cylindrical, or spherical), with r as co-ordinate direction, after Taylor series expansion left hand side of Eq. (1) and ignoring 2nd order and higher differential terms, we get

$$\Gamma\frac{\partial q}{\partial t}+q=-k\frac{\partial T}{\partial r} \tag{2}$$

with ρ as the density, c_p as the specific heat, and index $n=0$, 1 and 2 standing for planar, cylindrical and spherical geometry, in the absence of convection and radiation, in a generalized form, the governing energy equation for the 1-D geometry can be written as

$$\rho c_p\frac{\partial T}{\partial t}=-\nabla\cdot q=-\frac{1}{r^n}\frac{\partial(r^n q)}{\partial r} \tag{3}$$

From Eqs. (1) and (3), the governing energy equation in terms of dependant variable Tbecomes

$$\rho c_p\left(\Gamma\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}\right)=\frac{k}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right) \tag{4}$$

If the time lag $\Gamma=0$, Eq. (4) is the diffusion equation based on Fourier's law of heat conduction. With non-dimensional time ζ, distance η, temperature Θ, and heat flux Ψ defined as

$$\zeta=\frac{C^2\rho c_p t}{2k},\quad \eta=\frac{Cr}{2\alpha},\quad \Theta=\frac{TkC}{\alpha q_{ref}},\quad \Psi=\frac{q}{q_{ref}} \tag{5}$$

Eq. (6) becomes

$$\frac{\partial^2\Theta}{\partial\zeta^2}+2\frac{\partial\Theta}{\partial\zeta}=\frac{1}{\eta^n}\frac{\partial}{\partial\eta}\left(\eta^n\frac{\partial\Theta}{\partial\eta}\right) \tag{6}$$

In the LBM formulation, equation corresponding to Eq. (6) is written as

$$\begin{aligned} f_i(\eta+\mathbf{e}_i\Delta\zeta,\zeta+\Delta\zeta)-f_i(\eta,\zeta)&=-\frac{\Delta\zeta}{\tau}\left[f_i(\eta,\zeta)-f_i^{(0)}(\eta,\zeta)\right]\\ &-\frac{\Delta\zeta}{2}\left[\frac{n\left(a_i\vec{\eta}.\Psi\right)}{2\eta_i}+4b_i\mathbf{e}_i.\Psi\right], i=1,\ldots M \end{aligned} \tag{7}$$

where f_i is the particle distribution function, $f_i^{(0)}$ is the equilibrium particle distribution function, and for the D1Q2 lattice (Fig. 1), $\mathbf{e}_i=\frac{\Delta\vec{\eta}}{\Delta\zeta}(e_1=\frac{\Delta\eta}{\Delta\zeta}, e_2=-\frac{\Delta\eta}{\Delta\zeta})$ is the velocity with which f_i propagates to its nearest neighbor, τ is the relaxation time, Ψ is the heat flux, a_i and b_i are weights specific to the lattices used. In the D1Q2 lattice, the corresponding weights a_i, b_i and the relaxation time τ are given as

$$a_1=a_2=\frac{1}{2}; b_1=b_2=\frac{1}{2} \tag{8}$$

$$\tau=\left(\frac{\Delta\zeta}{\Delta\eta}\right)^2+\frac{\Delta\zeta}{2} \tag{9}$$

where $\Delta\zeta$ is the time step.

Solution of Eq. (7) requires knowledge of equilibrium distribution function $f_i^{(0)}$and heat flux Ψ. These are obtained from the following:

$$f_i^{(0)}=a_i\Theta+b_i\mathbf{e}_i\cdot\Psi \tag{10}$$

$$\Psi=\sum_i f_i\mathbf{e}_i=\sum_i f_i^{(0)}\mathbf{e}_i \tag{11}$$

Finally, with equilibrium distribution function f_i known from the solution of Eqs. (10) and (11), temperature distribution $\Theta(\eta,\zeta)$ is given by

$$\Theta=\sum_i f_i^{(0)}=\sum_i f_i \tag{12}$$

Eqs. (6) and (7) represent the governing equations of the same problem, in two different approaches, i.e., the macroscopic (continuum) and the mesoscopic (LBM) approaches, and they should provide the same results. The mesoscopic approach leads to the macroscopic. This equivalence is established through the Chapman–Enskog multi-scale expansion of Eq. (7) as shown in the following.

If the LHS of Eq. (7) is expanded we get

$$f_i(\eta+e_i\Delta\zeta,t+\Delta\zeta)=f_i(\eta,\zeta)+\Delta\zeta\frac{\partial f_i}{\partial\zeta}+\Delta\zeta\frac{\partial(e_i f_i)}{\partial\eta}+O\left(\Delta\zeta^2\right)+O(\Delta\eta^2) \tag{13}$$

With ε as the expansion parameter, with respect to the equilibrium particle distribution function $f_i^{(0)}$, f_i can be written as

$$f_i(r,t)=f_i=f_i^{(0)}+\varepsilon f_i^{(1)}+O(\varepsilon^2),\quad |\varepsilon|<<1 \tag{14}$$

Substituting Eq. (14) in the RHS of Eq. (7), we get

$$\begin{aligned}&\frac{\partial f_i^{(0)}}{\partial\zeta}+\varepsilon\frac{\partial f_i^{(1)}}{\partial\zeta}+\frac{\partial\left(e_i f_i^{(0)}\right)}{\partial\eta}+\varepsilon\frac{\partial\left(e_i f_i^{(1)}\right)}{\partial\eta}\\ &=-\frac{1}{\tau}\varepsilon f_i^{(1)}-\frac{\Psi}{2}\left[\frac{na_i}{2\eta_i}+4b_ie_i\right]+O(\Delta\zeta)+O(\varepsilon^2)\end{aligned} \tag{15}$$