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The characteristic variational multiscale method for time dependent conduction–convection problems☆,☆☆

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ABSTRACT

In this paper, the characteristic variational multiscale (C-VMS) method is proposed to solve the nonstationary conduction–convection problems. The stability analysis is carried out using the energy estimate method. Compared with the standard variational multiscale (VMS) method, the C-VMS method does not need nonlinear iteration. Finally, some numerical examples are given, which show that the C-VMS method is efficient, reliable and can save a lot of CPU time for this problem, besides, it can deal with the high Rayleigh number.

1. Introduction

In this paper, we consider the following time-dependent nonlinear conduction–convection equations with initial-boundary value problems:

$$\begin{cases} \mathrm{u}_t - \nu\Delta\mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla p = Ra\,\mathrm{j}T & \text{in } \Omega\times(0,T_1], \\ \nabla\cdot\mathrm{u} = 0 & \text{in } \Omega\times(0,T_1], \\ T_t - \nu\lambda\Delta T + \mathrm{u}\cdot\nabla T = \gamma & \text{in } \Omega\times(0,T_1], \\ \mathrm{u}(x,0) = \mathrm{u}^0, \quad T(x,0) = T^0 & \text{in } \Omega\times\{0\}, \\ \mathrm{u} = 0, \quad T = T_0 & \text{on } \partial\Omega\times(0,T_1], \end{cases} \tag{1}$$

where $\mathrm{u} = (u_1(\mathrm{x}), u_2(\mathrm{x}))$ represents the velocity vector, $p = p(\mathrm{x})$ the pressure, $T(\mathrm{x})$ the temperature, γ the forcing function, respectively. And $\mathrm{x} = (x_1, x_2)$, $\mathrm{j} = (0,,,1)^T$, $\lambda = \frac{1}{Pr}$, $\nu = \frac{1}{Re}$; Pr, Re, Ra, T_1 represent the Prandtl number, the Reynolds number, the Rayleigh number, and the given final time, respectively.

Conduction–convection problems are the same as natural convection problems whose control equations are composed of continuity equation, momentum equation, and energy equation. They constitute an important dissipative nonlinear equation in atmospheric dynamics. This is a hot topic in the heat transmission science for a long time, because it has been widely used in many fields of production and life. And the nonstationary conduction–convection problems are much more difficult than the stationary conduction–convection problems no matter if physical point of view or computation. Therefore, it is significant to study the problem and so far, many scholars have carried out important extensive research work.

In recent years, many scholars devoted a huge amount of research towards the development of these problems that can be found in literatures ([1,2,4,16,7] and the references therein). Boland and Layton [1] gave some numerical analyses and numerical results for the non-stationary natural convection equations. Luo and his collaborators offered lowest order finite difference scheme based on mixed finite element method (FEM) for non-stationary natural convection problem in [2]; what's more, they gave an optimizing reduced Petrov–Galerkin least squares mixed FEM combined with proper orthogonal decomposition method for non-stationary conduction–convection problems in [4]. In addition, Si and his collaborators [16] formulated the modified characteristic Gauge–Uzawa FEM for time dependent conduction–convection problems. And Trouette carried on the numerical simulation of Lattice Boltzmann method for time-dependent natural convection problem.

As is known, VMS methods which were firstly posed by Hughes in [8,9] are based on the decomposition of the flow scales and define the

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large scales by projection into appropriate subspaces. Then John et al. [10,11], Kaya and Rivière [12], Masud and Khurram [13], and Zheng et al. [14] studied and developed VMS methods. The characteristic methods have proved their efficiency for many physical problems, especially for convection-dominated problems (see Ref. [3]). A new VMS method is presented for steady-state natural convection problem with bubble stabilized FEM in [18]. In this paper, we shall use the C-VMS method to solve the nonstationary conduction–convection problems. The C-VMS method is derived from VMS method and the characteristics method, which is a highly effective method for time dependent conduction–convection problems and time dependent Navier–Stokes problems. Compared with VMS method which needs two iterations at least using Newton iterative at the same time layer, the C-VMS method does not need nonlinear iteration with the same accuracy. So the C-VMS method can save a lot of CPU time. Importantly, the C-VMS method can be used for the high Rayleigh number and the high Reynolds number.

The remainder of this paper is organized as follows. In Section 2, we will introduce some notations and give some preliminaries for problem (1). Then we present the C-VMS method for solving the time-dependent nonlinear conduction–convection problems in Section 3, and Section 4 is given stability analysis of the new scheme. Some numerical experiments conforming the theoretical results are provided in Section 5. Finally, conclusions are drawn in Section 6.

2. Preliminaries

For the mathematical setting of problem (1), we introduce the standard Hilbert spaces, finite element spaces and some notations.

$$\begin{aligned}
&\mathrm{X} = \mathrm{H}_0^1(\Omega)^2, \quad W = H^1(\Omega), \quad W_0 = H_0^1(\Omega),\\
&Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \quad dx = 0\},\\
&\mathrm{V} = \mathrm{H}_{0,\mathrm{div}}^1(\Omega)^2 = \{\mathrm{u} \in \mathrm{X}, \nabla\cdot\mathrm{u} = 0 \quad \text{in} \ \ \Omega\}.
\end{aligned}$$

Here and below, the space $L^2(\Omega)$ is equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. Further, we will consider the standard definitions for Sobolev spaces $W^{m,p}(\Omega)$ equipped with the norm $\|\cdot\|_{m,p}$ and semi-norm $|\cdot|_{m,p}$, $m, p \geq 0$. Note that

$$\|\cdot\|_m = \|\cdot\|_{m,2}, \quad |\cdot|_m = |\cdot|_{m,2}.$$

Then the weak formulation of Eq. (1) reads: seek $(\mathrm{u}, p, T) \in \mathrm{X}\times Q\times W$ for all $t \in (0, T_1]$ such that for all $(\mathrm{v}, q, s) \in \mathrm{V}\times Q\times W_0$ and $T|_{\partial\Omega} = T_0$,

$$\begin{cases}
(\mathrm{u}_t, \mathrm{v}) + B((\mathrm{u}, p); (\mathrm{v}, q)) + c(\mathrm{u}; \mathrm{u}, \mathrm{v}) = Ra(\mathrm{j}T, \mathrm{v}),\\
(T_t, s) + \overline{a}(T, s) + \overline{c}(u; T, s) = (\gamma, s),\\
\mathrm{u}(\mathrm{x}, 0) = \mathrm{u}^0, \quad T(\mathrm{x}, 0) = T^0,
\end{cases} \tag{2}$$

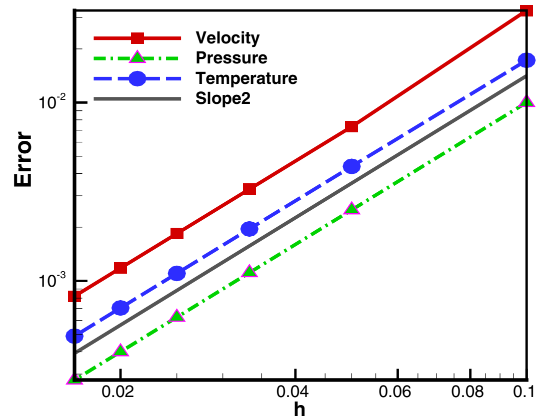

Fig. 1. Log–log errors of the velocity, pressure and temperature.

where

$$\begin{aligned}
&a(\mathrm{u}, \mathrm{v}) = \nu(\nabla\mathrm{u}, \nabla\mathrm{v}), b(\mathrm{v}, p) = (q, \nabla\cdot v), \overline{a}(T, s) = \lambda\nu(\nabla T, \nabla s),\\
&c(\mathrm{u}; \mathrm{v}, \mathrm{w}) = ((\mathrm{u}\cdot\nabla)\mathrm{v}, \mathrm{w}) + \frac{1}{2}((\nabla\cdot\mathrm{u})\mathrm{w}, \mathrm{v}) = \frac{1}{2}((\mathrm{u}\cdot\nabla)\mathrm{v}, \mathrm{w}) - \frac{1}{2}((\mathrm{u}\cdot\nabla)\mathrm{w}, \mathrm{v}),\\
&\overline{c}(u; T, s) = ((\mathrm{u}\cdot\nabla)T, s) + \frac{1}{2}((\nabla\cdot\mathrm{u})T, s) = \frac{1}{2}((\mathrm{u}\cdot\nabla)T, s) - \frac{1}{2}((\mathrm{u}\cdot\nabla)s, T),\\
&B((\mathrm{u}, p); (\mathrm{v}, q)) = a(\mathrm{u}, v) - b(\mathrm{v}, p) + b(\mathrm{u}, q).
\end{aligned}$$

Notes: with the above notations, there are the following estimates.

(A_1). The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition

$$\sup_{\mathrm{v}\in\mathrm{X}} \frac{b(\mathrm{v}, q)}{\|\nabla\mathrm{v}\|_0} \geq \beta\|q\|_0, \quad \forall\ q \in Q,$$

where β is a positive constant depending on Ω.

(A_2). As is known, the above trilinear forms $c(\cdot;\cdot,\cdot)$ and $\overline{c}(\cdot;\cdot,\cdot)$ have the following properties:

$$\begin{aligned}
&c(\mathrm{u}; \mathrm{w}, \mathrm{v}) = -c(\mathrm{u}; \mathrm{v}, \mathrm{w}),\\
&|c(\mathrm{u}; \mathrm{v}, \mathrm{w})| \leq N\|\nabla\mathrm{u}\|_0\|\nabla\mathrm{v}\|_0\|\nabla\mathrm{w}\|_0, \quad \forall \mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{X},
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&\overline{c}(\mathrm{u}; T, s) = -\overline{c}(\mathrm{u}; s, T),\\
&|\overline{c}(\mathrm{u}; T, s)| \leq \overline{N}\|\nabla\mathrm{u}\|_0\|\nabla T\|_0\|\nabla s\|_0, \quad \forall (\mathrm{u}, T, s) \in (\mathrm{X}, W, W),
\end{aligned} \tag{4}$$

where

$$N = \sup_{\mathrm{u},\mathrm{v},\mathrm{w}\in\mathrm{X}} \frac{|c(\mathrm{u}; \mathrm{v}, \mathrm{w})|}{\|\nabla\mathrm{u}\|_0\|\nabla\mathrm{v}\|_0\|\nabla\mathrm{w}\|_0}, \overline{N} = \sup_{\mathrm{u}\in\mathrm{X}; T, s\in W} \frac{|\overline{c}(\mathrm{u}; T, s)|}{\|\nabla\mathrm{u}\|_0\|\nabla T\|_0\|\nabla s\|_0}.$$

(A_3). The generalized bilinear form satisfies the continuity property and inf-sup condition [15]:

$$|B((\mathrm{u}, p); (\mathrm{v}, q))| \leq c(\|\nabla\mathrm{u}\|_0 + \|p\|_0)(\|\nabla\mathrm{v}\|_0 + \|q\|_0), \forall (\mathrm{u}, p), (\mathrm{v}, q) \in (\mathrm{X}, \mathrm{Q}),$$

Table 1
The convergence rates of C-VMS method with $\Delta t = h^2$.

$1/h$	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\frac{\|\nabla(T-T_h)\|_0}{\|\nabla T\|_0}$	K_{div}	CPU-time
10	2.8742E − 2	1.0000E − 2	1.7266E − 2	3.9433E − 3	2.37
20	7.3237E − 3	2.5001E − 3	4.3847E − 3	9.7962E − 4	28.52
30	3.2682E − 3	1.1111E − 3	1.9548E − 3	4.3398E − 4	137.62
40	1.8413E − 3	6.2507E − 4	1.1008E − 3	2.4360E − 4	437.33
50	1.1796E − 3	4.0011E − 4	7.0487E − 4	1.5570E − 4	1088.87
60	8.1988E − 4	2.7794E − 4	4.8963E − 4	1.0803E − 4	2211.15

Table 2
The convergence rates of C-VMS method with $\Delta t = 0.0001$.

$1/h$	$\frac{\|\nabla(u-u_h)\|_0}{\|\nabla u\|_0}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$\frac{\|\nabla(T-T_h)\|_0}{\|\nabla T\|_0}$	K_{div}	CPU-time
70	6.0194E − 4	2.0408E − 4	3.5979E − 4	7.9725E − 5	855.38
80	4.6093E − 3	1.5625E − 4	2.7550E − 4	6.1015E − 5	1136.88
90	3.6423E − 4	1.2345E − 4	2.1769E − 4	4.8194E − 5	1485.9
100	2.9505E − 4	9.9999E − 5	1.7634E − 4	3.9029E − 5	2140.78