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A novel finite volume method for cylindrical heat conduction problems☆

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ABSTRACT

A new finite volume method for cylindrical heat conduction problems based on Ln*r*-type diffusion equation is proposed in this paper with detailed derivation. On the basis of coordinate transformation, the diffusion term in the *r* direction of the heat conduction equation in a cylindrical coordinate is transformed into the Ln*r*-type diffusion term. Considering the influence of different boundary conditions, source terms and ratios of the internal to external radius, four typical categories of cases are calculated by the newly proposed method based on Ln*r*-type diffusion equation, the method based on local analytical solution and traditional central difference finite volume method respectively. The comparison of calculation results indicates that the new finite volume method is more accurate than the conventional one, since when the governing equation is discretized, the new method transforms $\lambda r \frac{\partial T}{\partial r}$ into $\lambda \frac{\partial T}{\partial \ln r}$ and treats this term as a whole to guarantee the conservativeness of diffusion flux. Numerical results also show that the total calculating time and required computation grid number of the new method are significantly less than the other two methods for achieving the same level of accuracy.

1. Introduction

To obtain the grid-independent solution under the same conditions, the computation grid number needed for solving the diffusion problems in cylindrical coordinate is larger than that in Cartesian coordinate due to the variation of diffusion area along the *r* direction [1]. Hence, considering the influence of such variation, Li et al. [1] proposed a finite volume method for cylindrical heat conduction problems based on local analytical solution and numerical calculations show that this method is more accurate than the central difference finite volume method, regardless of different boundary conditions, source terms and grids. And to reach the same precision, this method costs much less computation time than that of the traditional method.

However, the discretized expression of the method based on local analytical solution is far more complicated than that of the central difference finite volume method, and this imposes constrains in its applications. In order to overcome this drawback and ensure high calculation accuracy at the same time, the present paper proposes a new method based on Ln*r*-type diffusion equation which converts the diffusion equation in cylindrical coordinate into that in Cartesian coordinate by means of coordinate transformation.

In the following text, this new method is presented as follows. In Section 2, the derivation of Ln*r*-type diffusion equation in cylindrical coordinate is described in detail. Then abundant numerical cases are shown in Section 3 to compare the performance of the new finite volume method based on Ln*r*-type diffusion equation with that of the method based on local analytical solution and that of the traditional method, and illustrate the desirable features of the new method from different aspects. Finally, related conclusions are given in Section 4.

2. Derivation of Ln*r*-type diffusion equation in cylindrical coordinate

Firstly, it is necessary to briefly review the traditional central difference finite volume method and the finite volume method based on local analytical solution. The steady-state heat conduction equation in a cylindrical coordinate can be written as follows:

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda \frac{\partial T}{\partial r}\right) + S = 0. \quad (1)$$

The relationship between grid nodes and interfaces is shown in Fig. 1.

Discretizing the original governing equation over the control volume*P* (as shown in Fig. 1) by the central difference finite volume Scheme [2], Eq. (1) can be transformed and rearranged to the expression below:

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b \quad (2)$$

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Nomenclature

a_P,a_E,a_W,a_N,a_S coefficients in the discretized equation
b source term in the discretized equation
h_f convection heat transfer coefficient, W/(m^2 · °C)
k ratios of the internal to external radius, $k = r_1/r_2$
N_{Grid} total grid number
N_1 total grid number in the x–Lnr coordinate
N_2 total grid number in the x–r coordinate in reference [1]
N_3 total grid number in the x–r coordinate
q heat flux density, W/m^2
r spatial coordinate
r_1 internal radius,$r_1 = kr_2$, m
r_2 external radius, m
R non-dimensional coordinate in the r direction, $R = (r - r_1)/(r_2 - r_1)$
R_T ratio of the computation time
S heat source, W/m^3
S^* source term, $S^* = S + \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)$, W/m^3
t_1 computation time of Scheme 1 in Case 3(a)
t_2 computation time of Scheme 2 in Case 3(a)
t_3 computation time of Scheme 3 in Case 3(a)
T temperature, °C
T_f ambient temperature, °C
x spatial coordinate
x_l length of domain, m
X non-dimensional coordinate in the x direction, $X = x/l$

Greek symbols
λ thermal conductivity, W/(m · °C)
Δx, Δr width of control volume in the x and r direction
Δx, ΔLnr width of control volume in the x and Ln $r$direction
δx, δr, δ Ln r distance between adjacent nodes

Subscripts
e, w, n, s interfaces of the control volume P as shown in Fig. 1
P, E, W, N, S, NE, SE, NW, SW grid nodes as shown in Fig. 1

where

$$a_P = a_W + a_E + a_S + a_N$$
$$a_W = \frac{r_P\lambda_w\Delta r_P}{(\delta x)_w}, a_E = \frac{r_P\lambda_e\Delta r_P}{(\delta x)_e}, a_S = \frac{r_s\lambda_s\Delta x_P}{(\delta r)_s}, a_N = \frac{r_n\lambda_n\Delta x_P}{(\delta r)_n}$$
$$b = r_P\Delta x_P\Delta r_P S_P.$$

To improve the accuracy of numerical calculation, Li et al. [1] proposed a finite volume method based on local analytical solution as mentioned in Section 1. The heat flux q_n and q_s at the interface n and s can be expressed as follows:

$$q_n = \lambda\frac{\partial T}{\partial r}\bigg|_n = \frac{1}{r_n}\left(\frac{\frac{1}{2}\lambda_N S_P^*\left[r_n^2 \ln\frac{r_P}{r_n} + \frac{1}{2}\left(r_n^2 - r_P^2\right)\right] + \frac{1}{2}\lambda_P S_N^*\left[r_n^2 \ln\frac{r_n}{r_N} + \frac{1}{2}\left(r_N^2 - r_n^2\right)\right] + \lambda_P\lambda_N(T_N - T_P)}{\lambda_N \ln\frac{r_n}{r_P} + \lambda_P \ln\frac{r_N}{r_n}}\right) \tag{3}$$

$$q_s = \lambda\frac{\partial T}{\partial r}\bigg|_s = \frac{1}{r_s}\left(\frac{\frac{1}{2}\lambda_P S_S\left[r_s^2 \ln\frac{r_S}{r_s} + \frac{1}{2}\left(r_s^2 - r_S^2\right)\right] + \frac{1}{2}\lambda_S S_P\left[r_s^2 \ln\frac{r_s}{r_P} + \frac{1}{2}\left(r_P^2 - r_s^2\right)\right] + \lambda_S\lambda_P(T_P - T_S)}{\lambda_P \ln\frac{r_s}{r_S} + \lambda_S \ln\frac{r_P}{r_s}}\right). \tag{4}$$

Based on Eqs. (3) and (4), the discretized equation over the control volume P becomes:

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + b \tag{5}$$

where

$$a_P = a_E + a_W + a_N + a_S$$
$$a_W = \frac{r_P\lambda_w\Delta r_P}{(\delta x)_w}, a_E = \frac{r_P\lambda_e\Delta r_P}{(\delta x)_e}, a_S = \frac{\lambda_S\lambda_P\Delta x_P}{\lambda_P \ln\frac{r_s}{r_S} + \lambda_S \ln\frac{r_P}{r_s}},$$
$$a_N = \frac{\lambda_P\lambda_N\Delta x_P}{\lambda_N \ln\frac{r_n}{r_P} + \lambda_P \ln\frac{r_N}{r_n}}$$
$$b = \Delta x_P(b_1 - b_2) + r_P\Delta x_P\Delta r_P S_P$$
$$b_1 = \frac{\frac{1}{2}\lambda_N S_P^*\left[r_n^2 \ln\frac{r_P}{r_n} + \frac{1}{2}\left(r_n^2 - r_P^2\right)\right] + \frac{1}{2}\lambda_P S_N^*\left[r_n^2 \ln\frac{r_n}{r_N} + \frac{1}{2}\left(r_N^2 - r_n^2\right)\right]}{\lambda_N \ln\frac{r_n}{r_P} + \lambda_P \ln\frac{r_N}{r_n}}$$
$$b_2 = \frac{\frac{1}{2}\lambda_P S_S^*\left[r_s^2 \ln\frac{r_S}{r_s} + \frac{1}{2}\left(r_s^2 - r_S^2\right)\right] + \frac{1}{2}\lambda_S S_P^*\left[r_s^2 \ln\frac{r_s}{r_P} + \frac{1}{2}\left(r_P^2 - r_s^2\right)\right]}{\lambda_P \ln\frac{r_s}{r_S} + \lambda_S \ln\frac{r_P}{r_s}}.$$

Taking interface n as a boundary, the temperature at the boundary can be obtained for the second and third boundary conditions respectively as follows.

For the second boundary condition:

$$q_n = \lambda\frac{\partial T}{\partial r}\bigg|_n = q_B \tag{6}$$

$$T_N = \frac{q_B r_n \ln\frac{r_n}{r_P} - \frac{1}{2}S_P^*\left[r_n^2 \ln\frac{r_P}{r_n} + \frac{1}{2}\left(r_n^2 - r_P^2\right)\right]}{\lambda_P} + T_P. \tag{7}$$

For the third boundary condition:

$$q_n = \lambda\frac{\partial T}{\partial r}\bigg|_n = h_f\left(T_f - T_N\right) \tag{8}$$

$$T_N = \frac{\lambda_P T_P + \ln\frac{r_n}{r_P} r_n h T_f - \frac{1}{2}S_P^*\left[r_n^2 \ln\frac{r_P}{r_n} + \frac{1}{2}\left(r_n^2 - r_P^2\right)\right]}{r_n h_f \ln\frac{r_n}{r_P} + \lambda_P}. \tag{9}$$

The results of eight numerical cases in the literature [1] show that, to reach the same precision, the computation time and required grid number of the method based on local analytical solution are less than that of the traditional central difference finite volume method [1]. These advantageous features can attribute to the use of Eqs. (3) and (4) to calculate heat flux at interface which involves the significant influence of diffusion area and source term along the r direction. But the complicated discretized process of this method brings about great difficulties in subsequent programming, thus the present paper proposes a simple and convenient method to calculate heat flux precisely which deals with $\lambda r \frac{\partial T}{\partial r}$ as a whole by means of coordinate transformation. On the basis of $\partial(\ln r) = \frac{\partial r}{r}$, the heat flux at interface which is $\lambda r \frac{\partial T}{\partial r}$ can be