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Second order fully discrete and divergence free conserving scheme for time-dependent conduction–convection equations ☆,☆☆

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ABSTRACT

This work presents a Crank–Nicolson extrapolation scheme for the two-dimensional time-dependent conduction–convection equations. Mixed finite element method is applied for the spatial approximation of the velocity, pressure and temperature. The time discretization is based on the Crank–Nicolson scheme for the linear term and semi-implicit scheme for the nonlinear term. Moreover, the stability analysis and error estimations are derived. Finally, numerical tests confirm the theoretical results of the presented method and show the efficient method conserves the property of divergence free of the original equations to some extent.

1. Introduction

The nonstationary conduction–convection problems constitute an important system of equations in atmospheric dynamics and a dissipative nonlinear system of equations. Since this system of equations does not only contain the velocity field as well as the pressure field but also contain the temperature field, finding the numerical solutions of this problem is a difficult task. Thus, development of an efficient computational method for investigating this problem has practical significance, and has drawn the attention of many researchers. At the time of writing, there are numerous works devoted to the development of efficient schemes for the conduction–convection equations [1–8].

In general, there exist fully implicit, semi-implicit (semi-explicit), and explicit scheme to deal with the time-dependent problems. Among them, high-order schemes are of more interest because first-order schemes are not sufficiently accurate for large time approximations. Meanwhile, the stability condition of schemes is also a key issue. Usually an explicit scheme is much easier in computation. But it suffers a severely restricted time step size from stability requirement. A fully implicit scheme is (almost) unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. Hence, a popular approach is based on an implicit scheme for the linear term and a semi-implicit scheme or an explicit scheme for the nonlinear term. A semi-implicit scheme for the nonlinear term results in a linear system with a variable coefficient matrix of time, and an explicit treatment for the nonlinear term gives a constant matrix.

Especially, the well known Crank–Nicolson extrapolation scheme is one of these approaches. Moreover, the scheme is a second-order accuracy which is based on the Crank–Nicolson scheme and is very popular to deal with the nonstationary equations. Currently, the Crank–Nicolson extrapolation scheme is applied to the time discretization of the Navier–Stokes equations by Girault and Raviart [9] and Simo and Armero [10]. Also, the Crank–Nicolson extrapolation scheme is applied to the time discretization of the nonlinear parabolic equations (see Douglas and Dupont [11], Cannon and Lin [12], and Lin [13]) and the nonlinear dynamics (see Simo, Tarnow, and Wong [14]). Moreover, He and Sun [15] have provided an error analysis for the Crank–Nicolson extrapolation scheme of time discretization applied to the spatially discrete stabilized finite element approximation of the two-dimensional time-dependent Navier–Stokes problem, and the low-order finite element have been applied.

This article focuses on the Crank–Nicolson extrapolation scheme for the two-dimensional nonstationary conduction–convection equations.

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The second-order fully discrete scheme is based on the Crank–Nicolson extrapolation scheme, in which we use an implicit scheme for the viscous and pressure terms and a semi-implicit scheme for the nonlinear term. Mixed finite element method (e.g. P_2-P_1-P_1, P_1b-P_1-P_1, etc.) is applied for the spatial approximation of the velocity, pressure and temperature. Then the numerical results illustrated the efficiency of the proposed scheme and in a way the scheme guarantees the property of solenoidal vector field for the original problem.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations, an abstract functional setting of the conduction–convection problem. Mixed finite element strategy is recalled and some well-known results are used throughout this paper in Section 3. Second-order fully discrete method based on Crank–Nicolson extrapolation scheme is given in Section 4. Then in Section 5, numerical experiments are shown to verify the theoretical results completely. Finally, we end with a short conclusion in Section 6.

2. Preliminaries

Let Ω be a bounded, convex and open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. We consider the time-dependent conduction–convection equations:

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \lambda jT, & \text{in} \quad \Omega\times(0,\mathrm{T}_1], \\ \nabla\cdot u = 0, \quad \text{in} \quad \Omega\times(0,\mathrm{T}_1], \\ T_t - \Delta T + \lambda u\cdot\nabla T = 0, \quad \text{in} \quad \Omega\times(0,\mathrm{T}_1], \\ u(x,0) = 0, \quad T(x,0) = 0, \quad \text{on} \quad \Omega\times\{0\}, \\ u = 0, \quad T = T^0 \quad \text{in} \quad \partial\Omega\times(0,\mathrm{T}_1], \end{cases} \tag{1}$$

where $u = (u_1(x, t), u_2(x, t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $T = T(x,t)$ the temperature, $\lambda > 0$ the Grashoff number, $j = (0, 1)^T$ the two-dimensional vector, $\nu > 0$ the viscosity, T_1 the given final time and $u_t = \partial u/\partial t$, $T_t = \partial T/\partial t$.

With the standard Sobolev spaces

$$X = \left(H_0^1(\Omega)\right)^2, \quad W = H^1(\Omega), \quad W_0 = H_0^1(\Omega),$$
$$M = L_0^2(\Omega) = \left\{q\in L^2(\Omega) : \int_\Omega q dx = 0\right\},$$

a weak formulation of Eq. (1) reads: find $(u, p, T) \in (X, M, W)$ for all $t \in (0, T_1)$ such that for all $(v, q, s) \in (X, M, W_0)$ and $T|_{\partial\Omega} = T^0$,

$$\begin{cases} (u_t, v) + B((u,p);(v,q)) + b(u;u,v) = \lambda(jT, v), \\ (T_t, s) + \bar{a}(T,s) + \lambda\bar{b}(u;T,s) = 0, \\ u(x,0) = 0, \quad T(x,0) = 0, \end{cases} \tag{2}$$

with

$$a(u,v) = \nu(\nabla u, \nabla v), \quad d(v,q) = (q, \mathrm{div} v), \quad \bar{a}(T,s) = (\nabla T, \nabla s),$$
$$b(u;v,w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\mathrm{div} u)w, v) = \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v),$$
$$\bar{b}(u;T,s) = ((u\cdot\nabla)T, s) + \frac{1}{2}((\mathrm{div} u)T, s) = \frac{1}{2}((u\cdot\nabla)T, s) - \frac{1}{2}((u\cdot\nabla)s, T),$$
$$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q).$$

The trilinear forms $b(\cdot;\cdot,\cdot)$ and $\bar{b}(\cdot;\cdot,\cdot)$ satisfy

$$\begin{aligned} |b(u;v,w)| &\le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0, \quad \forall u,v,w\in X, \\ \left|\bar{b}(u;T,s)\right| &\le \bar{N}\|\nabla u\|_0\|\nabla T\|_0\|\nabla s\|_0, \quad \forall (u,T,s)\in(X,W,W), \end{aligned} \tag{3}$$

where

$$N = \sup_{u,v,w\in X}\frac{|b(u;v,w)|}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0}, \quad \bar{N} = \sup_{u\in X, T,s\in W}\frac{\left|\bar{b}(u;T,s)\right|}{\|\nabla u\|_0\|\nabla T\|_0\|\nabla s\|_0}.$$

3. Mixed finite element method

For $h > 0$, we introduce finite-dimensional subspaces $(X_h, M_h, W_h) \subset (X, M, W)$ which are characterized by K_h, a partitioning of Ω into triangle K with the mesh size h, assumed to be uniformly regular in the usual sense. We also define $W_{0h} = W_h \cap W_0$. For further details, readers can refer to Ciarlet [16]. Subsequently, c (with or without a subscript) will denote a generic positive constant.

The standard finite element Galerkin approximation of Eq. (2) based on (X_h, M_h, W_h) reads as follows: find $(u_h, p_h, T_h) \in (X_h, M_h, W_h)$ such that, for all $0 \le t \le T_1$, $T_h|_{\partial\Omega} = T_h^0$ (T_h^0 is the interpolation of T^0) and $(v_h, q_h, s_h) \in (X_h, M_h, W_{0h})$,

$$\begin{cases} (u_{ht}, v_h) + B((u_h,p_h);(v_h,q_h)) + b(u_h;u_h,v_h) = \lambda(jT_h, v_h), \\ (T_{ht}, s_h) + \bar{a}(T_h,s_h) + \lambda\bar{b}(u_h;T_h,s_h) = 0, \\ u_h(x,0) = 0, \quad T_h(x,0) = 0. \end{cases} \tag{4}$$

Then, we define the subspace V_h of X_h given by

$$V_h = \{v_h\in X_h : d(v_h,q_h) = 0, \forall q_h\in M_h\}.$$

We know from [17] that the pair (X_h, M_h) and V_h satisfy the following approximation properties:

Lemma 3.1. *Let $I_h : L^2(\Omega)^2 \to V_h$ be the standard L^2-projection. Then*

$$\|v - I_h v\|_{0,\Omega} + h\|\nabla(v - I_h v)\|_{0,\Omega} \le ch^i\|v\|_{i,\Omega}, \quad \forall v\in H^i(\Omega)\cap V_0, \tag{5}$$

for $i = 1,2,3$, *with* $V_0 = \{v \in H_0^1(\Omega); \nabla\cdot v = 0\}$.

Purely for some subspace analysis, we shall often make use of the approximate divergence-free finite element space V_{0h}:

$$V_{0h} = \{v_h\in V_h; (\mathrm{div} v_h, q_h) = 0, \forall q_h\in M_h\}.$$

From the results of [8,16,17], the pair (X_h, M_h, W_h) and V_{0h} satisfy the following properties:

P(A). For each $v \in H^i(\Omega) \cap H_0^1(\Omega)$ with $\nabla\cdot v = 0$ and $q \in H^{i-1} \cap L_0^2(\Omega)$ with $i = 1,2,3$, there exist approximations $\pi_h v \in V_{0h}$ and $\rho_h q \in M_h$ such that

$$\|\nabla(v - \pi_h v)\|_0 \le ch^{i-1}\|v\|_i, \quad \|q - \rho_h q\|_0 \le ch^{i-1}\|q\|_{i-1}.$$

P(B). There exists r_h: $W \to W_h$, such that for all φ

$$(\nabla(\varphi - r_h\varphi), \nabla\varphi_h) = 0, \quad \forall\varphi_h\in W_h,$$
$$\int_\Omega(\varphi - r_h\varphi)dx = 0, \quad \|\nabla r_h\varphi\|_0 \le \|\nabla\varphi\|_0,$$

and that when $\varphi \in W^{r,q}(\Omega)(1 \le q \le \infty)$, have

$$\|\varphi - r_h\varphi\|_{-s,q} \le ch^{r+s}|\varphi|_{r,q}, \quad -1\le s\le m, 0\le r\le m+1. \tag{6}$$

P(C). The inverse inequality:

$$\|\nabla v_h\|_0 \le ch^{-1}\|v_h\|_0, \quad \forall v_h\in X_h.$$

P(D). For each $q_h \in X_h$, there exists $v_h \in X_h$, $v_h \neq 0$ such that

$$d(v_h,q_h) \ge \beta\|q_h\|_0\|\nabla v_h\|_0,$$

here β is a positive constant depending only on Ω.

Then, we need a further assumption on T_0 proved in [8].