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Determination of a time-dependent thermal diffusivity and free boundary in heat conduction☆

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ABSTRACT

In this paper, we consider the inverse problem of simultaneous determination of time-dependent leading coefficient (thermal diffusivity) and free boundary in the one-dimensional time-dependent heat equation. The resulting inverse problem is recast as a nonlinear regularized least-squares problem. Stable and accurate numerical results are presented and discussed.

1. Introduction

Many heat transfer applications can be modeled by the heat equation with a fixed boundary. However, there are numerous other problems for which the domain or the boundary varies with time and such problems are known as free boundary or Stefan problems [1]. For instance, when a conductor melts and the liquid is drained away as it appears, the heat conduction problem within the remaining solid involves the heat equation in a domain that is physically changing with time. In particular, the one-phase Stefan problem can be regarded as an inverse problem.

In [2], the author investigated the heat equation with an unknown heat source in a domain with a known moving boundary. In [3,4], the authors investigated the numerical solution of inverse Stefan problems using the method of fundamental solutions. In [5], an inverse moving boundary problem is solved using the least-squares method. In our work we consider the time-dependent nonlinear inverse one-dimensional and one-phase Stefan problem which consists of the simultaneous determination of the time-dependent thermal diffusivity and free boundary.

This paper is organized as follows: In the next section, we give the formulation of the inverse problem under investigation. The numerical methods for solving the direct and inverse problems are described in Sections 3 and 4, respectively. Furthermore, the numerical results and discussion are given in Section 5 and finally, conclusions are presented in Section 6.

2. Mathematical formulation

Consider the one-dimensional time-dependent heat equation

$$\frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \qquad (x,t) \in \Omega \tag{1}$$

in the domain $\Omega = \{(x,t): 0 < x < h(t), 0 < t < T < \infty\}$ with unknown free smooth boundary $x = h(t) > 0$ and time-dependent thermal diffusivity $a(t) > 0$. The initial condition is

$$u(x,0) = \phi(x), \quad 0 \le x \le h(0) =: h_0, \tag{2}$$

where $h_0 > 0$ is given, and the boundary and over-determination conditions are

$$u(0,t) = \mu_1(t), \quad u(h(t),t) = \mu_2(t), \quad 0 \le t \le T, \tag{3}$$

$$-a(t)u_x(0,t) = \mu_3(t), \quad \int_0^{h(t)} u(x,t)dx = \mu_4(t), 0 \le t \le T. \tag{4}$$

Note that μ_1 and μ_3 represent Cauchy data at the boundary end $x = 0$, while μ_4 represents the specification of the energy of the heat conducting system, [6].

First we perform the change of variable $y = x/h(t)$ to reduce the problem (1)–(4) to the following inverse problem for the unknowns $a(t)$, $h(t)$ and $v(y,t) := u(yh(t),t)$:

$$\frac{\partial v}{\partial t}(y,t) = \frac{a(t)}{h^2(t)}\frac{\partial^2 v}{\partial y^2}(y,t) + \frac{yh'(t)}{h(t)}\frac{\partial v}{\partial y}(y,t) + f(yh(t),t), \quad (y,t) \in Q \tag{5}$$

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in the fixed domain $Q=\{(y,t):0<y<1, 0<t<T\}$ with unknown time-dependent coefficients $a(t)$ and $h(t)$. The initial condition is

$$v(y,0)=\phi(h_0 y), \quad 0\le y\le 1, \tag{6}$$

and the boundary and over-determination conditions are

$$v(0,t)=\mu_1(t), \quad v(1,t)=\mu_2(t), \quad 0\le t\le T, \tag{7}$$

$$-a(t)_y(0,t)=\mu_3(t)h(t), \quad h(t)\int_0^1 v(y,t)dy=\mu_4(t), 0\le t\le T. \tag{8}$$

This model has been considered in [7]. The triplet $(h(t),a(t),v(y,t))$ is called a solution to the inverse problem (5)–(8) if it belongs to the class $C^1[0,T]\times C[0,T]\times C^{2,1}\left(\overline{Q}\right)$, $h(t)>0$, $a(t)>0$, $t\in[0,T]$ and satisfies Eqs. (5)–(8). For the input data we make the following regularity and compatibility assumptions:

(A) $\mu_i(t)\in C^1[0,T], \mu_i(t)>0$ for $t\in[0,T]$, $i=1,2,4$, $\mu_3(t)\in C^1[0,T]$, $\mu_3(t)<0$ for $t\in[0,T]$, $\phi(x)\in C^2[0,h_0]$, $\phi(x)>0$, $\phi'(x)>0$ for $x\in[0,h_0]$, and $f(x,t)\in C^{1,0}([0,H_1]\times[0,T])$, $f(x,t)\ge 0$ for $(x,t)\in[0,H_1]\times[0,T]$, where

$$H_1=\max_{[0,T]}\mu_4(t)\left(\min\left\{\min_{[0,h_0]}\phi(x),\min_{[0,T]}\mu_1(t),\min_{[0,T]}\mu_2(t)\right\}\right)^{-1};$$

(B) $\phi(0)=\mu_1(0)$, $\phi(h0)=\mu_2(0)$, and $\int_0^{h0}\phi(x)dx=\mu_4(0)$.

The following existence and uniqueness of solution theorems are proved in [7].

Theorem 1. (Local existence)

If the conditions (A) and (B) are satisfied, then there exists $t_0\in[0,T]$, (defined by the input data) such that a solution of problem (5)–(8) exists locally for $(y,t)\in[0,1]\times[0,t_0]$.

Theorem 2. (Uniqueness)

Suppose that the following conditions are satisfied:

(i) $0\le f(x,t)\in C^{1,0}([0,H_1]\times[0,T])$;
(ii) $\phi(x)>0$ *for* $x\in[0,h_0]$, $\mu_1(t)>0, \mu_2(t)>0, \mu_3(t)<0$, *and* $\mu_4(t)>0$ *for* $t\in[0,T]$.

Then a solution to problem (5)–(8) is unique.

3. Solution of direct problem

In this section, we consider the direct initial boundary value problem (5)–(7), where $a(t)$, $h(t)$, $f(x,t)$, $\in(x)$, and $\mu_i(t)$, $i=1,2$, are known and the solution $u(x,t)$ is to be determined additionally with $\mu_i(t)$, $i=3,4$. To achieve this, we use the Crank–Nicolson finite-difference scheme [8], which is unconditionally stable and second-order accurate in space and time.

The discrete form of our problem is as follows. We divide the domain $Q=(0,1)\times(0,T)$ into M and N subintervals of equal step length Δy and Δt, where $\Delta y=1/M$ and $\Delta t=T/N$, respectively. So, the solution at the node (i,j) is $v_{i,j}:=v(y_i,t_j)$, where $y_i=i\Delta y$, $t_j=j\Delta t$, and $a(t_j)=a_j$, $h(t_j)=h_j$ and $f(y_i,t_j)=f_{i,j}$ for $i=\overline{0,M}$, $j=\overline{0,N}$. Based on the Crank–Nicolson method, Eq. (5) can be approximated as:

$$\begin{aligned}&-A_{i,j+1}v_{i+1,j+1}+\left(1+B_{j+1}\right)v_{i,j+1}-C_{i,j+1}v_{i-1,j+1}\\&=A_{i,j}v_{i+1,j}+\left(1-B_j\right)v_{i,j}+C_{i,j}v_{i-1,j}+\frac{\Delta t}{2}\left(f_{i,j}+f_{i,j+1}\right)\end{aligned} \tag{9}$$

for $i=\overline{1,(M-1)}$, $j=\overline{0,N}$, where

$$A_{i,j}=\frac{(\Delta t)\alpha_j}{2(\Delta y)^2}-\frac{(\Delta t)\gamma_j y_i}{4\Delta y}, \quad B_j=\frac{(\Delta t)\alpha_j}{(\Delta y)^2}, \quad C_j=\frac{(\Delta t)\alpha_j}{2(\Delta y)^2}+\frac{(\Delta t)\gamma_j y_i}{4\Delta y},$$
$$\alpha_j=\frac{a_j}{h_j^2}, \quad \gamma_j=\frac{h'\left(t_j\right)}{h_j}.$$

The initial and boundary conditions (6) and (7) can also be collocated as:

$$v_{i,0}=\phi(h_0 y_i), \quad i=\overline{0,M}, \tag{10}$$

$$v_{0,j}=\mu_1\left(t_j\right), \quad v_{M,j}=\mu_2\left(t_j\right), \quad j=\overline{0,N}. \tag{11}$$

At each time step t_j, for $j=\overline{0,(N-1)}$, using the Dirichlet boundary conditions (11), the above difference Eq. (9) can be reformulated as a $(M\text{-}1)\times(M\text{-}1)$ system of linear equations of the form,

$$L\mathbf{u}=\mathbf{b}, \tag{12}$$

where

$$\mathbf{u}=\left(v_{1,j+1},v_{2,j+1},\ldots,v_{M-1,j+1}\right)^{tr}, \quad \mathbf{b}=\left(b_1,b_2,\ldots,b_{M-1}\right)^{tr}$$

and

$$L=\begin{pmatrix}1+B_{j+1} & -C_{1,j+1} & 0 & \cdots & 0 & 0 & 0\\ -A_{2,j+1} & 1+B_{j+1} & -C_{2,j+1} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -A_{M-2,j+1} & 1+B_{j+1} & -C_{M-2,j+1}\\ 0 & 0 & 0 & \cdots & 0 & -A_{M-1,j+1} & 1+B_{j+1}\end{pmatrix},$$

$$b_1=A_{1,j}v_{0,j}+\left(1-B_j\right)v_{1,j}+C_{1,j}v_{2,j}+A_{1,j+1}v_{0,j+1}+\frac{\Delta t}{2}\left(f_{1,j+1}+f_{1,j}\right),$$
$$b_i=A_{i,j}v_{i-1,j}+\left(1-B_j\right)v_{i,j}+C_{i,j}v_{i+1,j}+\frac{\Delta t}{2}\left(f_{i,j+1}+f_{i,j}\right), \quad i=\overline{2,(M-2)},$$
$$\begin{aligned}b_{M-1}&=A_{M-1,j}v_{M-2,j}+\left(1-B_j\right)v_{M-1,j}+C_{M-1,j}v_{M,j}+C_{M-1,j+1}v_{M,j+1}\\&+\frac{\Delta t}{2}\left(f_{M-1,j+1}+f_{M-1,j}\right).\end{aligned}$$

As an example, consider the problem (5)–(7) with $T=1$ and

$$a(t)=1+t, \quad h(t)=1+2t, \quad h_0=h(0)=1, \quad \phi(h_0 y)=(1+y)^2,$$
$$\mu_1(t)=1+8t, \quad \mu_2(t)=(2+2t)^2+8t, \quad f(h(t)y,t)=6-2t.$$

The exact solution of the direct problem (5)–(7) is given by $v(y,t)=(1+y+2yt)^2+8t$, and the desired outputs are $\mu_3(t)=-2(1+t)$ and $\mu_4(t)=\frac{(2+2t)^3-1}{3}+8t(1+2t)$. The numerical and exact solutions for $v(y,t)$ are shown in Fig. 1 and very good agreement is obtained. Tables 1 and 2 give the numerical heat flux at $y=0$ and the numerical integral in comparison with the exact values, i.e. μ_3 and μ_4. These have been calculated using the following $O(h^2)$ finite-difference approximations for derivative and trapezoidal rule for integration:

$$v_y\left(0,t_j\right)=\frac{4v_{1,j}-v_{2,j}-3v_{0,j}}{2\Delta y}, \qquad j=\overline{1,N}, \tag{13}$$

$$\int_0^1 v\left(y,t_j\right)dy=\frac{\Delta y}{2}\left(v\left(0,t_j\right)+v\left(1,t_j\right)+2\sum_{i=1}^{M-1}v\left(y_i,t_j\right)\right), \qquad j=\overline{0,N}. \tag{14}$$