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Approximate analytic solution of heat conduction in hollow semi-spheres flying at hypersonic speed ☆

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ABSTRACT

Heat transfer in blunt noses of hypersonic vehicles with coolant inside can be approximately considered as heat conduction in hollow semi-sphere with aerodynamic heating on the outer boundary and enhanced cooling on the inner boundary. Theoretical investigations of temperature field in hollow semi-spheres were carried out by solving the two-dimensional axsymmetric conduction equation, which could be transformed into Legendre equation when the separation of variables is applied. However, for such a semi-sphere flying at hypersonic speed, the distribution of heat transfer rates as an outer boundary condition is so complex that the integration in the Legendre solution is nearly impossible to be completed. In this paper, a 4th order Legendre polynomial, derived by the method of undetermined coefficients, was adopted to approach the local similarity solution of hypersonic aerodynamic heating and simplify the integration process, by which an approximate solution could be set up for the temperature field. The approximate solution is also validated by comparing the analytical results with data from numerical simulations, in which the conduction equation is solved with the improved Richardson scheme. Both analytical and numerical results are compared to each other and match quite well.

1. Introduction

For hypersonic vehicles, serious aerodynamic heating is a great barrier. Both heat transfer rates at local positions and their integration over surface and flight time are considered to be governing factors for the design of thermal protection systems (TPS) [1]. Usually, it is difficult for researchers to get a clear figure on such a physical phenomenon due to the complexity of aerodynamic heating by hypersonic flow and conduction inside the structure. During the last two decades, numerical simulations have led to some better understanding of physical mechanisms and applications in engineering, such as the coupling of aerodynamic heating and structural conduction, enhanced convective cooling et al. However, as effective ways for scientific researches, it is usually not easy for numerical simulations or experiments to show the regularity between parameters, such as temperature field, flow field, dimension, and heat transfer rates et al. In the latest decade, plenty of investigations were dedicated to regenerative cooling of hypersonic vehicles, and the most typical examples are thermal protection systems for propulsion systems, such as scramjets [2]. The aerodynamic heating or heat transfer in engines due to combustion will result in high heat flux into structure. If the structural temperature went too high, the performance of material, structure, and surface coating, such as oxidation resistance, strength, cooling efficiency et al., will get significant discounts, which may result in serious damage to the vehicles. Usually, for regenerative cooling, fuel is used as coolant to absorb the heat and lower the temperature of structure in hypersonic air-breathing vehicles. During the cooling process, fuel gets preheated and may lead to higher combustion efficiency due to phase changing and fuel cracking happening during the fuel heating.

In this paper, the model was set as a hollow semi-sphere flying at hypersonic speed with convective cooling inside, as shown in Fig. 1(a). Thereby, the heat conduction in the semi-sphere is the main subject in the following chapters. To simplify the investigations, the high temperature environment generated by the hypersonic coming flow is replaced by an aerodynamic heating distribution on the outer boundary, which is an analytical form governed by the local similarity solution [3]. By assuming the inner convective cooling inside with efficiency high enough, the temperature of the inner wall is much lower than the critical value, which may result in discount of material and structural strength, oxidation resistance, or ablation of surface coating. Therefore, the governing equation and ideal model can be shown as Eq. (1), which is also depicted in Fig. 1(b)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=0 \quad (1)$$

According to the hypothesis made before, the boundary conditions for Eq. (1) can be shown as follows

$$\left.\frac{\partial T}{\partial\theta}\right|_{\theta=0}=\left.\frac{\partial T}{\partial\theta}\right|_{\theta=\pi/2}=0 \quad (2)$$

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Nomenclature

A_n, B_n	coefficients of series or polynomial
$f(\theta)$	normalized heat rate on outer boundary
$g(\theta)$	approximate normalized heat rate on outer boundary
M, M_∞	Mach number of hypersonic incoming flow
P_n	first class Legendre function
Q_n	second class Legendre function
q_w	aerodynamic heat rate distribution on outer boundary
r, θ	axis of spherical coordinate system
r_1,r_2	inner and outer radius of hollow semi-sphere
T	temperature field of hollow semi-sphere
T_n	series form of T
T_w	temperature distribution on inner boundary
x,y	axis of Cartesian coordinate system
n	natural number
i	No. of the grid point for the finite difference method

Greek letters

Δt	time step
Δx	space step
Γ	Γ function
α	ratio between r_2 and r_1
β	coefficient of parabolic equation
ε	circumferential motion factor
ξ, η	axis of computational space coordinate system
γ	Euler number
γ_∞	specific heat ratio of incoming flow

$$T\Big|_{r=r_1} = T_w(\theta) \tag{3}$$

$$\frac{\partial T}{\partial r}\Big|_{r=r_2} = q_w(\theta) \tag{4}$$

Since the solution of Eq. (1) will be in form of the Legendre function [4], and the integration process is very difficult to operate, this paper will find out an approximate analytical solution by using a 4th order Legendre polynomial to approach the local similarity solution for aerodynamic heating in hypersonic flight, which makes the integration process in the original analytical solution possible. Further, the governing equations with boundary conditions as depicted in Eqs. (2) to (4) was numerically solved by a finite difference method with the improved Richardson scheme. Both analytical and numerical solutions match with each other, so the approximate analytical solution can be considered as demonstrated and validated. Details and further discussions about the applications of such a solution will be represented in the following chapters.

2. Theoretical solution

The solution of Eqs. (1) to (4) is not very complex, which might be found in some textbooks or reference books. For the convenience of understanding and checking such a paper, a few key procedure of the derivation process will be included in the following. With separation of variables, the temperature field can be defined as $T(r,\theta)=R(r)\Phi(\theta)$, and Eq. (1) could be transformed in to a typical eigenvalue problem with two equations shown as follows

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta R\frac{d\Phi}{d\theta}\right)+\lambda\Phi=0 \tag{5 − a}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\lambda R=0 \tag{5 − b}$$

where $\lambda=n(n+1)$, and $n=0,1,2\ldots$

Eqs. (5-a) and (5-b) can be solved individually with linear superposition for the final solution. If the transformations as $\tau=\cos\theta$ and $\varphi(\tau)=\Phi(\theta)$ were applied and substituted into Eq. (5-a), and the following equation could be obtained

$$\frac{d}{d\tau}\left[\left(1-\tau^2\right)\frac{d\varphi}{d\tau}\right]+n(n+1)\varphi=0 \tag{6}$$

This equation is classic Legendre equation, and the generalized solution in the neighborhood of $\tau=1(\theta=0)$ can be stated as

$$\varphi(\tau)=C_1P_n(\tau)+C_2Q_n(\tau) \tag{7}$$

where $P_n(\tau)$ and $Q_n(\tau)$ represent the first and second class of Legendre polynomials, respectively. The detailed expressions are listed as following

$$P_n(\tau)=\sum_{k=0}^{n}\frac{1}{(k!)^2}\frac{\Gamma(n+k+1)}{\Gamma(n-k+1)}\left(\frac{\tau-1}{2}\right)^k \tag{8 − a}$$

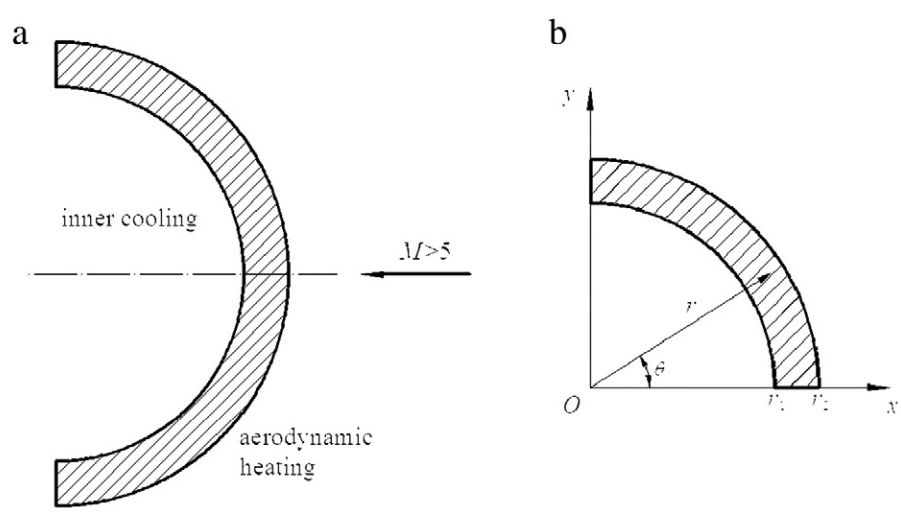

Fig. 1. Physical phenomenon and modeling: (a) schematic of blunt nose of hypersonic vehicles with coolant inside; (b) schematic of computational domain for heat conduction.