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# Low Prandtl number chaotic convection in porous media with uniform internal heat generation ☆

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ABSTRACT

Using the theory of dynamical systems, this study investigated the effects of a uniform internal heat generation on chaotic behaviour in thermal convection in a fluid-saturated porous layer subject to gravity and heated from below for low Prandtl number. A low-dimensional, Lorenz-like model was obtained using Galerkin truncated approximation. The fourth-order Runge–Kutta method was employed to solve the nonlinear system. We found that there is an inverse proportional relation between the level of internal heat $G$ and the scaled Rayleigh number $R$, and consequently the porous media gravity-related Rayleigh number $Ra$.



## 1. Introduction

Chaotic behaviour in a fluid-saturated porous medium has attracted interest due to its wide application in such fields as geothermal energy utilization, oil reservoir modeling, catalytic packed beds filtration, thermal insulation and nuclear waste disposal [1].

Vadasz and Olek [2] found that the transition from steady convection to chaos is sudden and occurs by a subcritical Hopf bifurcation producing a solitary limit cycle which may be associated with a homoclinic explosion when the Prandtl number is low. This finding can be recovered from a truncated Galerkin expansion [3] that yields a system identical to the familiar Lorenz equations [4,5]. The work of Vadasz [6] suggests an explanation for the appearance of this solitary limit cycle via local analytical results. A similar approach was used by Vadasz [7] to demonstrate similar results for the corresponding convection problem in a pure fluid. Vadasz and Olek [8] showed that, when the Prandtl number is moderate, the route to chaos occurs by a period doubling sequence of bifurcations.

An understanding of the effects of internal heat generation, on the other hand, is important in several applications that include reactor safety analysis, metal waste, spent nuclear fuel, fire and combustion studies and strength of radioactive materials [9].

Islam [10] investigated natural convection in both horizontal and inclined porous channels with uniform internal heat generation. He observed chaotic behaviour with increasing Rayleigh number. Fluid flow and heat transfer in a porous medium over a stretching surface with internal heat generation was investigated by Rafael [11], who observed that with increasing Prandtl number the temperature decreases. Mealey and Merkin [12] considered steady convective flow within a square region filled with a fluid-saturated porous medium having internal heat generation at a rate proportional to the power of the temperature difference. They found that the flow and heat transfer depended on the Rayleigh number and a heat generation parameter, as well as the local-heating exponent.

In this study, the work of Vadasz and Olek [2] on the transition to chaos is extended to include consideration of the influence of a uniform internal heat generation. The transition from steady convection to chaos was analyzed using the fourth-order Runge–Kutta method. On the grounds that it is important to understand low-dimensional dynamics before moving to more complex systems, truncated Galerkin approximation was applied to the governing equations for thermal convection in a fluid-saturated porous layer subject to gravity and heated from below for a low Prandtl number, allowing us to deduce an autonomous system with three ordinary differential equations. This system was used to investigate the dynamic behaviour of thermal convection in a porous medium and to elucidate the effects of internal heat generation on the transition to chaos.

## 2. Problem formulation

Consider a fluid-saturated porous layer subject to gravity and heated from below with local heat generation. A Cartesian co-ordinate system is used such that the vertical axis $z$ is collinear with gravity, i.e. $\hat{e}_g = \hat{e}_z$.

The dimensionless governing equations can be written as

$$\nabla . \vec{q} = 0, \tag{1}$$

$$\left[\frac{1}{Va}\frac{\partial}{\partial t} + 1\right]\vec{q} = -\nabla p + RaT\hat{e}_z, \tag{2}$$

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**Nomenclature**

| | |
|---|---|
| $Da$ | Darcy number, defined by $k_*/L_*^2$ |
| $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$ | unit vectors in the $x$, $y$ and $z$-directions |
| $\hat{e}_g$ | unit vector in the direction of gravity |
| $\hat{e}_n$ | unit vector normal to the boundary, positive outwards |
| $G$ | a measure of the internal heat generation |
| $H_*$ | the height of the layer |
| $H$ | the front aspect ratio of the porous layer, equals $H_*/L_*$ |
| $k_*$ | permeability of the porous domain |
| $L_*$ | the length of the porous layer |
| $L$ | reciprocal of the front aspect ratio, equals $1/H = L_*/H_*$ |
| $M_f$ | a ratio between the heat capacity of the fluid and the effective heat capacity of the porous domain |
| $p$ | reduced pressure (dimensionless) |
| $Pr$ | Prandtl number, equals $\nu_*/\alpha_e^*$ |
| $q$ | dimensionless filtration velocity vector, equals $u\hat{e}_x + v\hat{e}_y + w\hat{e}_z$ |
| $Ra$ | porous media gravity-related Rayleigh number, equals $\beta_*\Delta T_c g_* H_* k_* M_f/\alpha_e^*\nu_*$ |
| $R$ | scaled Rayleigh number, equals $Ra/4\pi^2$ |
| $\hat{R}$ | $(4\gamma)R/(4\gamma - b)$ |
| $t$ | time |
| $T$ | dimensionless temperature, equals $(T_* - T_C)/(T_H - T_C)$ |
| $T_C$ | coldest wall temperature |
| $T_H$ | hottest wall temperature |
| $u$ | horizontal $x$ component of the filtration velocity |
| $v$ | horizontal $y$ component of the filtration velocity |
| $Va$ | Vadasz number, equals $\phi Pr/Da$ |
| $w$ | vertical component of the filtration velocity |
| $x, y, z$ | Cartesian co-ordinates |
| $X$ | rescaled amplitude $A_{11}$ |
| $Y$ | rescaled amplitude $B_{11}$ |
| $Z$ | rescaled amplitude $B_{02}$ |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | a parameter related to the time derivative term in Darcy's equation |
| $\alpha_e^*$ | effective thermal diffusivity |
| $\beta_*$ | thermal expansion coefficient |
| $\gamma$ | $L/\theta = L^2/(L^2+1)$ |
| $\phi$ | porosity |
| $\nu_*$ | fluid's kinematic viscosity |
| $\mu_*$ | fluid's dynamic viscosity |
| $\Psi$ | stream function |
| $\Delta T_c$ | characteristic temperature difference |
| $\tau$ | rescaled time, equals$(L^2+1)\pi^2 t/L^2$ |
| $\theta$ | $(L^2+1)/L$ |

*Subscripts*

| | |
|---|---|
| * | dimensional values |
| c | characteristic state |
| cr | critical values |

$$\frac{\partial T}{\partial t} + \vec{q}\cdot\nabla T = \nabla^2 T + GT, \tag{3}$$

where $\vec{q}$ is the velocity, $T$ is temperature, $p$ is pressure, $Ra$ is the gravity-related Rayleigh number defined in the form $Ra = \beta_*\Delta T_c g_* H_* k_* M_f/\alpha_{e^*}\nu_*$, $Va$ is the Vadasz number defined by $Va = \phi Pr/Da$, and $G$ is a measure of internal heat generation. The time derivative term was included in Darcy's Eq. (2). The values $\alpha_e{}^*/H_* M_f$, $\mu_*\alpha_{e^*}/k_* M_f$, and $\Delta T_c = (T_H - T_C)$ were used to scale the filtration velocity components ($u_*$, $v_*$, $w_*$), pressure ($p_*$), and temperature variations ($T_* - T_C$), respectively, where $\alpha_e^*$ is the effective thermal diffusivity, $\mu_*$ is fluid viscosity, $k_*$ is the permeability of the porous matrix, and $M_f$ is the ratio between the heat capacity of the fluid and the effective heat capacity of the porous domain. The height of the layer $H_*$ was used for scaling the variables $x_*$, $y_*$, $z_*$ and $H_*^2/\alpha_e{}^*$ for scaling the time $t_*$. Accordingly, $x = x_*/H_*$, $y = y_*/H_*$, $z = z_*/H_*$ and $t = t_*\alpha_e{}^*/H_*^2$.

As all the boundaries are rigid, the solution must follow the impermeability conditions on the boundaries, that is, $\vec{q}\cdot\hat{e}_n = 0$, where $\hat{e}_n$ is a unit vector normal to the boundary. The temperature boundary conditions are $T(0) = 1$, $T(1) = 0$ and $\nabla T\cdot\hat{e}_n = 0$ on the other two vertical walls, representing the insulation condition on these walls.

The governing equations can be presented in terms of a stream function defined by $u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$, as for convective rolls having axes parallel to the shorter dimension (i.e. $y$) when $v = 0$. Applying the curl $(\nabla\times)$ operator on Eq. (2) yields the following system of partial differential equations from Eqs. (1)–(3):

$$\left[\frac{1}{Va}\frac{\partial}{\partial t} + 1\right]\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right] = -Ra\frac{\partial T}{\partial x}, \tag{4}$$

$$\frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} + GT, \tag{5}$$

where the boundary conditions for the stream function are $\psi = 0$ on all solid boundaries. The set of partial differential Eqs. (4) and (5), form a nonlinear coupled system and together with the corresponding boundary conditions will accept a basic motionless conduction solution.

## 3. Reduced set of equations and analysis

In order to obtain the solution to the nonlinear coupled system of partial differential equations in Eqs. (4) and (5), we represent the stream function and temperature in the form

$$\psi = A_{11}\sin\left(\frac{\pi x}{L}\right)\sin(\pi z), \tag{6}$$

$$T = \cos\left(\sqrt{G}z\right) - \cot\left(\sqrt{G}\right)\sin\left(\sqrt{G}z\right) + B_{11}\cos\left(\frac{\pi x}{L}\right)\sin(\pi z) + B_{02}\sin(2\pi z). \tag{7}$$

This representation is equivalent to a Galerkin expansion of the solution in both the $x$- and $z$-directions, truncated when $i + j = 2$, where $i$ is the Galerkin summation index in the $x$-direction and $j$ is the Galerkin summation index in the $z$-direction. Substituting Eqs. (6) and (7) into Eqs. (4) and (5), multiplying the equations by the orthogonal eigenfunctions corresponding to Eqs. (6) and (7) and integrating them over the domain, that is, $\int_0^L dx\int_0^1 dz(\cdot)$, yields a set of three ordinary differential equations for the time evolution of the amplitudes:

$$\frac{\partial A_{11}}{\partial\tau} = -\frac{Va\gamma}{\pi^2}\left[A_{11} + \frac{Ra}{\pi\theta}B_{11}\right], \tag{8}$$

$$\frac{\partial B_{11}}{\partial\tau} = \left(\frac{\gamma G}{\pi^2} - 1\right)B_{11} + \frac{4\pi}{\theta(G - 4\pi^2)}A_{11} - \frac{1}{\theta}A_{11}B_{02}, \tag{9}$$

$$\frac{\partial B_{02}}{\partial\tau} = \left(-4\gamma + \frac{\gamma G}{\pi^2}\right)B_{02} + \frac{1}{2\theta}A_{11}B_{11}, \tag{10}$$