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An efficient green element formulation for the Graetz problem ☆

Okey Oseloka Onyejekwe

Department of Mechanical Engineering, Northern Arizona University, Flagstaff, AZ 86011, United States

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ABSTRACT

We propose an efficient Green element method (GEM) technique for the solution of the generalized Graetz problem. The main point is to illustrate how GEM concepts can be adapted to handle heat or mass transport in tube flow; with axial conduction first ignored but later included. Several numerical examples are tested to demonstrate this numerical approach; for all cases, it is seen that GEM offers an elegant and in comparison with the problem difficulty, a reliable and straightforward approach.

1. Introduction

Because of its importance, the Graetz problem, is still an area of active research. Earlier work on the solution of the Graetz problem was based on investigations of thermally developing flow for cases of forced heat transfer in a circular pipe with various simplifying assumptions. Several researchers have presented analytical solutions to this problem by converting the eigenvalue problem into a system of ordinary differential equations in which the eigenvalues are determined (Barron et al. [1]). Numerous strategies including the introduction of a rapidly converging series solutions were developed to facilitate the determination of more eigenvalues with greater accuracy (Sellars et al. [2]). Adopting the method of Papoutsakis et al. [3] for laminar pipe flow, Weigand [4] came up with an entirely analytical solution to the extended turbulent Graetz problem with Dirichlet wall boundary conditions. His solution method was found to be applicable for a broad class of problems described by the same governing equation. Foucher and Mansour [5] employed a novel approximate analytical solution for laminar forced convection in parallel plate ducts subjected to periodic variations of inlet temperatures with time. Their work neglected axial heat conduction along the walls but took into account the transverse temperature gradient inside the walls as well as fluid-to-solid heat capacitance ratio effect on the periodic inlet temperature. To verify their proposed analytical Galerkin technique, they solved the same problem using a numerical finite difference method, and later compared the results. An excellent account of related work can be found in Shah and Bhatti [6], as well as Guedes and Ozisik [7].

Although conventional boundary element techniques have been used in the past to solve transport problems, such as those by Brebbia and Skerget [8], Taigbenu and Liggett [9], and Cheng [10], there is a noticeable scarcity of reference to the application of BEM for the numerical solution of the generalized Graetz problem, especially for cases where axial conduction is included. The difficulty in obtaining an analytical Green's function power law-type axial velocity profiles applicable to this type of problem as well as the inability to deal efficiently with the problem domain must have accounted for this. To address this problem, Ramachandran [11] developed a methodology which applied boundary-element concepts at intervals or elements in the radial direction rather than for the entire domain of integration. In order to solve the resulting integral equation, he approximated the variation of the dependent variable by a cubic polynomial within each subdomain. The equations were then given in terms of the nodal variables. Considerable effort was made to eliminate the radial diffusion operator by the application of some specialized weighting functions. There was no attempt to deal with cases involving axial conduction.

Recent modifications of BEM focus on the fact that the problem domain holds the key to converting BEM into a highly efficient and robust numerical technique. This objective has been explored via two different routes. First as a *boundary driven approach* (Nardini and Brebbia [12], Popov and Powers [13], Dargush and Banerjee [14]); and as a hybrid BEM–FEM *domain driven approach* (Taigbenu and Onyejekwe [15], Onyejekwe [16]). Application of these methods has become more widespread because of the far more efficient accurate results obtained in applying them to problems that have sometimes proved intractable to the standard BEM technique.

Several numerical experiences with BEM have led to encounters with domain dominant problems for which the standard BEM is at a disadvantage in comparison with domain based techniques. This has motivated the need to modify the BEM in such a way that domain integration could be better addressed. One of such methods, the DRM-MD integral equation method (Popov and Powers [13]) transforms domain integrals into boundary integrals at the contour of each subregion, by using the Dual Reciprocity Method (DRM). While this method goes a step further than the DRM (Nardini and Brebbia [12]) in dividing the problem domain into several subregions or elements, it stops short of actually implementing cell integration, and instead it still keeps faith with BEM methodology by transferring all such integrations to the boundary of the problem domain. On the contrary, GEM departs from the DRM-MD

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E-mail address: okuzaks@yahoo.com.

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approach by embarking on a FEM type domain discretization. As a result it avoids much of the limitations attributed to the DRM. This has paved the way to significantly extending the range of application of BEM.

The work reported herein deals with the application of a novel GEM formulation to a variation of the classical Graetz problem involving heat transfer in two spatial directions one along the axis and the other in the direction perpendicular to the flow direction in annular or axisymmetric cylindrical channels. This problem finds a number of industrial applications involving heat transfer through annular channels, tubular reactors and flow in a circular tube with a constant wall heat flux. GEM's robustness is demonstrated by how efficiently the solutions are obtained.

2. Problem formulation

If axial conduction is neglected, the general problem under consideration can be represented by the following differential equation (Ramachandran [11])

$$\rho c_p u(r)\frac{\partial T}{\partial x} = \frac{k}{r^v}\frac{\partial}{\partial r}(r^v) + M_1 T + M_2 \tag{1}$$

where ρ is the fluid density, c_p is the specific heat of fluid, x is the axial coordinate, r is the radial coordinate and is perpendicular to flow direction, k is the thermal conductivity, $v = 0$, 1, 2, to represent 1D diffusion operator for an infinite slab, axisymmetric cylinder or sphere (for this application $v = 1$). The term M_1T represents a linear volumetric source, while M_2 is a constant source term. A simple power law is used for representing the local axial velocity at position r (Ramachandran [17])

$$u(r) = u_{max}\left[1 - \left(\frac{r}{R}^{\frac{(n+1)}{n}}\right)\right] \tag{2}$$

where R is the pipe radius, n is the power law index of flow field and u_{max} is the maximum velocity at $r = 0$, and is given by

$$u_{max} = u_{av}\left(\frac{n + 1 + n(\alpha + 1)}{n + 1}\right) \tag{3}$$

where u_{av} is the average velocity of fluid. After the chores of non-dimensionalization, Eq. (1) becomes

$$\left(1 - y^{\beta}\right)\frac{\partial\phi}{\partial\theta} = \frac{1}{y^v}\frac{\partial}{\partial y}(y^v) + M\phi + B_r \tag{4}$$

where θ, a time like variable, is a dimensionless axial distance : $x/R(k/\rho c_p u_{max}R)$, y is a dimensionless radial coordinate, $\beta = (n+1)/n$, M is a dimensionless heat generation term (M_1R^2/k), φ is the dimensionless temperature (T/T_{ref}), and B_r is the Brinkmann number $(k_2R^2/T_{Ref}k)$. The following boundary conditions specify Eq. (4)

(i) At entry of tube$(\theta = 0)$, $\varphi = 1.0$.
(ii) At$y = 0$ $\quad M\varphi/My = 0$.
(iii) At$y = 1$ $\quad a_1M\varphi/My + a_2\varphi = a_3(\theta)$. (5)

3. Green element solution method

Our main task here is to transform Eq. (4) into an integral form. This is achieved via the Green's second identity. We initiate this step by introducing into Eq. (4) a complementary equation

$$\nabla^2 G = \delta(x - x_i) \tag{6a}$$

whose solution is given by

$$G = \frac{(|x - x_i| + L)}{2} \tag{6b}$$

where L is an arbitrary constant and is set to represent the longest element in the problem domain. The discretized version of the governing equation is:

$$\begin{aligned}&2\lambda_i\phi(y_i,t) + (H(y_2 - y_i) - H(y_i - y_2))\phi_2\\ &\quad -(H(y_1 - y_i) - H(y_i - y_1))\phi_1\left((|y_2 - y_i|) + \bar{l}\right)\psi_2 - \left(\left(|y_1 - y_i| + \bar{l}\right)\right)\psi_1\\ &\quad + \int_{y_1}^{y_2}\left(|y - y_i| + \bar{l}\right)\left[\left(1 - y^{\beta}\right)\frac{\partial\phi}{\partial\theta} - \left(\frac{1}{y}\frac{\partial\phi}{\partial y}\right) - M\phi - B_r\right]dy = 0\end{aligned} \tag{7}$$

where $H(x)$ is the Heaviside function, and ψ is the spatial derivative of the dependent variable, λ is a parameter which specifies the location of the *source node* by using the properties of the *dirac delta* function. We mention in passing here that the GEM formulation uses the fundamental solution of the highest order of derivative term of the governing equation. This has been found to be a very utilitarian approach because it remains the same for different types of problems including linear nonlinear and heterogeneous problems. The solution of Eq. (7) on each element of the problem domain emphasizes GEM's finite element component. For example, expressing Eq. (7) on a typical 2-node element results in the following compact matrix equation:

$$R_{ij}\phi_j + P_{ij}\psi_j + T_{ij}\left(\left(1 - y^{\beta}\right)\frac{\partial\phi}{\partial\theta} - M\phi - B_r\right) - \int_{y_1}^{y_2}\frac{1}{y}\frac{\partial\phi}{\partial y}G(y,y_i)dy = 0 \tag{8a}$$

where

$$Z_{ij} = \int_{y_1}^{y_2}\frac{S_j}{y}G(y,y_i). \tag{8b}$$

Details from earlier work (Onyejekwe [15]) shows how Eq. (8a) can be represented as:

$$R_{ij}\varphi_j + \left(P_{ij}\right)\psi_j + T_{ij}\left(\left(1 - y^{\beta}\right)\frac{\partial\varphi}{\partial\theta} - M\varphi - B_r\right) - Z_{ij}\frac{\partial\phi}{\partial y} = 0. \tag{8c}$$

Eq. (8c) can be expressed differently by making the flux term instead of the spatial derivative of the scalar variable one of the dependent variables. As a result, Eq. (8c) can now be expressed as

$$R_{ij}\phi_j - \left(P_{ij}\right)\zeta_j + T_{ij}\left(\left(1 - y^{\beta}\right)\frac{\partial\phi}{\partial\theta} - M\phi - B_r\right) = 0 \tag{8d}$$

where $\zeta = -q/k$. The coefficients of Eq. (8d) have been described elsewhere (Onyejekwe [16]), except Z_{ij} which is given by

$$\begin{aligned}Z_{11} &= \int_{y_1}^{y_2}\left[\left(\frac{\Omega_1}{y}\right)G(y,y_1)\right]dy = \frac{1}{\Delta y}\int_{y_1}^{y_2}\left[\left(\frac{(y_1 - y)}{y}\right)\left(y - y_1 + \bar{l}\right)\right]dy\\ &= \frac{(y_2 - y_1)}{2} - \bar{l} + \frac{y_1\left(\bar{l} - y_1\right)}{\Delta y}[\ln y_2 - \ln y_1]\end{aligned} \tag{9a}$$

$$\begin{aligned}Z_{12} &= \int_{y_1}^{y_2}\left[\left(\frac{\Omega_2}{y}\right)G(y,y_1)\right]dy = \frac{1}{\Delta y}\int_{y_1}^{y_2}\left[\left(\frac{(y - y_1)}{y}\right)\left(y - y_1 + \bar{l}\right)\right]dy\\ &= \frac{(y_2 - 3y_1)}{2} + \bar{l} + \frac{y_1\left(y_1 - \bar{l}\right)}{\Delta y}[\ln y_2 - \ln y_1]\end{aligned} \tag{9b}$$